ユニタリ表現論セミナー報告集 IX, 1989
pp.66-75

Szegö Operators and a Paley-Wiener Theorem on SU(1,1)

《Abstract》

Takeshi KAWAZOE (Keio Univ.)

[1] Introduction.

The Paley-Wiener theorem on G=SU(1,1), that is, the characterization of the class of functions which are Fourier transforms of compactely supported, C^∞ functions on SU(1,1) was first obtained by Ehrenpreis and Mautner in 1955. In this talk I will give a new and simple proof as an application of the classical Szegö operators.

Before the explanation of the approach we shall recall in brief the proof given by Ehrenpreis and Mautner. The surjectivity of the Paley-Wiener theorem, that is, showing compactness of the supports of functions on G whose Fourier transforms are holomorphic functions of exponential type, is a difficult part of the proof. In order to obtain the surjectivity they used changing of contour of integration in the Fourier inversion formula, as originally Paley and Wiener used in the case of the Euclidean space. After Harish-Chandra's work of shperical functions and c-functions, in 1974 Johnson rephrased this method by using generalized c-functions. Then the main obstacles they encountered in their proof were the following: (1) how to obtain a sharp estimate for the Harish-Chandra expansion of the spherical functions which allows us to change the contour of integration, and (2) how to treat residues which appear during the contour change.

When we treat right or left K-invariant functions on G, the above residues don't appear during the contour change. On the other hand, when we treat K-finite functions on G, we encounter the residues. As shown by Ehrenpreis and Mautner, these residues are deeply related to L^2 matrix coefficients of nonunitary principal series of G, that is, reflected a realization of discrete series as a subrepresentation of the nonunitary principal series. In general, we can say that analysis of the residues which appear during contour change is nothing but analysis of subrepresentations of nonunitary principal series. This was done by Arthur in 1983 for arbitrary reductive Lie groups.

The aim of the new approach I will explain in this talk is to avoid this calculation of the residues. If we could reduce the proof of the Paley-Wiener theorem for K-finite functions to the one for right K-invariant functions, we hope that the theorem will be solved without treating the residues, in other words, if we could prove the theorem for K-finite functions without using the Harish-Chandra's generalized c-functions, we will not encountor the poles of c-functions and need not calculate their residues. In what follows, noting the classical Szegö operators, I shall give the proof of the Paley-Wiener theorem on SU(1,1) in this direction.

[2] Fourier Transform.

The Plancherel formula for $L^2(G)$ implies that each L^2

function on G can be written as the sum of wave packets and a linear combination of cusp forms, so $L^2(G)$ has a direct sum decomposition:

$$L^2(G) = {}^{P}L^2(G) \oplus {}^{\circ}L^2(G).$$

In what follows we shall restrict our attention to L^2 functions on G with right K-type n ($n \in \frac{1}{2}Z$) and left K-type is of free. Then the above decomposition is denoted by

$$L^2_n(G) = {}^{P}L^2_n(G) \oplus {}^{\circ}L^2_n(G).$$

Now we shall define the Fourier transforms on G associated with the principal and discrete series of G as follows.

Let $\varepsilon \in \{0, \frac{1}{2}\}$, $n \in Z_\varepsilon$ and $\nu = \frac{1}{2} + i\lambda \in C$. For f in $C_c^\infty(G)$ we define the Fourier transform $f^\wedge(\lambda, \zeta)$ $((\lambda, \zeta) \in R \times T)$ associated with the principal series $(\pi_{\varepsilon,\nu}, L^2(T))$ by

$$f^\wedge(\lambda, \zeta) = \int_G f(g) \mathrm{conj}(\pi_{\varepsilon,\nu}(g) e_{n-\varepsilon}(\zeta)) dg,$$

where $\zeta \in T$ and $e_{n-\varepsilon}$ is a unit vector in $L^2(T)$ with K-type n.

Let $n \in Z_\varepsilon$ and $I_n = \{\ell \in Z_\varepsilon \; ; \; 1 \leqq \ell \leqq n\}$. Then for $m \in I_n$ and $f \in C_c^\infty(G)$ we define the Fourier transform $F_{-m}(f)(z)$ $(z \in D)$ associated with the discrete series $(T_{-m}, A_{2,m-1}(D))$ by

$$F_{-m}(f)(z) = \int_G f(g) \mathrm{conj}(T_{-m}(g) {}^{\circ}e_{n-m}(z)) dg,$$

where $z \in D$ and ${}^{\circ}e_{n-m}$ is a unit vector in $A_{2,m-1}(D)$ with K-type n. Let $f^\wedge(z)$ denote a vector of functions on D given by

$$f^\wedge(z) = (F_{-m}(f)(z) \ ; \ m \in I_n)$$

Then the Fourier transform for $L^2(G)$ is finally defined by

$$f^\wedge = (f^\wedge(\lambda, \zeta), \ f^\wedge(z)) \quad ((\lambda, \zeta, z) \in R \times T \times D).$$

The image of the Fourier transforms of $L^2_n(G)$ is characterized by the direct sum of the following two spaces: the first one is

$$L^2_n(R \times T) = \{ \alpha(\lambda, \zeta) \in L^2(R \times T, \mu_\varepsilon(\lambda) d\lambda d\zeta) \ ;$$

$$A_n(\alpha)(\lambda, x) = \int_T \pi_{\varepsilon, 1-\nu^-}(x) e_{n-\varepsilon}(\zeta) \alpha(\lambda, \zeta) d\zeta \ \text{ is } \quad \text{even} \}$$

with obvious norm and the second one is the direct sum of the weighted Bergman spaces:

$$A^2_n(D) = \bigoplus_{m \in I_n} A_{2, m-1}(D)$$

with norm given by the sum of each norm. Then noting the original Plancherel formula for $L^2(G)$ stated in [EM], we can easily rephrase it as follows.

<u>Theorem 1</u>. The Fourier transform $f \to f^\wedge$ is an isometry of $L^2_n(G)$ onto $L^2_n(R \times T) \oplus A^2_n(D)$.

[3] Szegö Operators.

Now we shall give the definition of the Szegö operators, and then rephrase the inversion formula in Theorem 1.

First we shall define two line bundles:

$$C^\infty(K,\sigma_\varepsilon)=\{f\in C^\infty(K);\ f(mk)=\sigma_\varepsilon(m)f(k)\ \text{for}\ m\in M,\ k\in K\}$$

and

$$C^\infty(G,\tau_n)=\{f\in C^\infty(G);\ f(gk_\theta)=\tau_n(k_\theta)f(g)\ \text{for}\ k_\theta\in K,\ g\in G\},$$

where $\sigma_\varepsilon\in M^\wedge$ and $\tau_n\in K^\wedge$. We identify $C^\infty(T)$ with $C^\infty(K,\sigma_\varepsilon)$ by the operator defined by $I_\varepsilon(F)(k_\theta)=e^{i\varepsilon\theta}F(e^{i\theta})$, actually, I_ε is an isometry between $L^2(T)$ and $L^2(K,\sigma_\varepsilon)$, the L^2 completion of $C^\infty(K,\sigma_\varepsilon)$.

Then for $\nu\in C$ the Szegö operator

$$S_{\varepsilon,\nu,n}:C^\infty(K,\sigma_\varepsilon)\to C^\infty(G,\tau_n)$$

is defined by

$$S_{\varepsilon,\nu,n}(f)(x)=\int_K e^{\nu H(x^{-1}k)}\tau_n(\kappa(x^{-1}k))f(k)dk$$

$$=e^{in(\theta+\theta')}(1-|w|^2)^{-\nu}$$

$$\times\int_0^{2\pi}\frac{|1-e^{-i\psi}w|^{2n}}{(1-e^{-i\psi}w)^{2n}}|1-e^{-i\psi}w|^{2\nu}e^{-in\psi}f(k_\psi)d\psi,$$

where $x=k_\theta a_t k_{\theta'}\in G$ and $w=x\cdot 0=\text{tht}/2e^{i\theta}\in D$ (cf. [KW], p.178).

Clearly, $S_{\varepsilon,\nu,n}(f)\equiv 0$ except $n\in Z_\varepsilon$, and when $\varepsilon=\frac{1}{2}$, $\nu=-\frac{1}{2}$ and $n=\pm\frac{1}{2}$, the integral part of $S_{\frac{1}{2},-\frac{1}{2},\pm\frac{1}{2}}(f)(x)$ coincides with the classical Szegö projection operator on $L^2(T)$ (cf. [R], p.178). Actually, for $F\in L^2(T)$ with the Fourier series $\Sigma_{p\in Z}\, a_p e^{ip\theta}$, if we let

$$F_+(w) = \Sigma_{p=0}^{\infty} a_p w^p \quad (w\in D),$$

then

$$S_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}}(I_\varepsilon(F))(x) = e^{\frac{1}{2}in(\theta+\theta')}(1-|w|^2)^{\frac{1}{2}} F_+(w)$$

and

$$S_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}(I_\varepsilon(F))(x) = e^{-\frac{1}{2}i(\theta+\theta')}(1-|w|^2)^{\frac{1}{2}} F_+(\bar{w}).$$

[4] Inversion Formula.

By using the Szegö operators defined in [3] we shall rewrite the inversion formula in Theorem 1.

For $\gamma=(\alpha,\beta)\in L^2_n(R\times T)\oplus A^2_n(D)$, where $\beta=(\beta_m)$, we let

$$\gamma^\vee(x) = \alpha^\vee(x) + \beta^\vee(x)$$

$$= \int_R S_{\varepsilon,-\nu,n}(I_\varepsilon\alpha(\lambda,\cdot))(x)\,\mu_\varepsilon(\lambda)d\lambda$$

$$+ \sum_{m\in I_n} (\Gamma(2m)/\Gamma(n-m+1)\Gamma(n+m))^{\frac{1}{2}} E_+^{n-m}\, \beta_m^\vee(x),$$

where

$$\beta_m^\vee(x)=(4\pi)^{-1}(2m-1)(1-|w|^2)^m e^{im(\theta+\theta')}\beta_m(w)$$

and E_+ is the differential operator on G defined by

$$E_+ = d\pi_{\varepsilon,\nu}(X) + i\,d\pi_{\varepsilon,\nu}(Y).$$

In particular, if $\beta(z)$ has a bounded boundary value on T, then the second sum can be rewritten as (see Theorem 3 below)

$$\sum_{m \in I_n} (4\pi)^{-1}(2m-1)(\lambda_{n-m}/\lambda_{n-\varepsilon})^2\, S_{\varepsilon,m-1,n}(\lambda_{n-m}{}^{-1} I_\varepsilon \beta e_{\varepsilon-m})(x).$$

Then, since $E_+ S_{\varepsilon,\nu,n} = (n-\nu) S_{\varepsilon,\nu,n+1}$, we can easily deduce the following proposition. This is the merit of the expression using the Szegö operators.

<u>Proposition 2</u>. Let f be in ${}^{P}C^{\infty}_{c,n}(G)$. Then

(1) $f(x) = (E_+)^{n-\varepsilon} \int_R S_{\varepsilon,-\nu,\varepsilon}(I_\varepsilon f^\wedge(\lambda, \cdot) P_n(\lambda)^{-1})\, \mu_\varepsilon(\lambda)\, d\lambda,$

where $P_n(\lambda) = (n-½+i\lambda)(n-3/2+i\lambda) \cdots (\varepsilon+½+i\lambda).$

(2) $A_\varepsilon(f^\wedge P_n{}^{-1})(\lambda, x) = A_\varepsilon(f^\wedge P_n{}^{-1})(-\lambda, x) \quad ((\lambda, x) \in R \times G).$

Here we note that the integral appeared in (1) is nothing but apply the inversion formula for $L^2_\varepsilon(G)$ to the function $f^\wedge(\lambda, \zeta) P_n(\lambda)^{-1}$ which satisfies the functional equation in (2). As mentioned before, the Plancherel formula for $L^2_\varepsilon(G)$ is simpler than one for $L^2_n(G)$, because it is made up only of wave packets, that is, the discrete part does not appear. So, by this reduction we hope the proof of the Paley-Wiener theorem for $C^\infty_{c,n}(G)$ can be deduced to one for $C^\infty_{c,\varepsilon}(G)$.

Of course, in this reduction the poles of $P_n(\lambda)^{-1}$ will be a new obstacle when we directly use the Paley-Wiener theorem for $C^\infty_{c,\varepsilon}(G)$. However, as we show in [5], we need not calculate the residues of the poles of $P_n(\lambda)^{-1}$.

[5] A New Proof of The Paley-Wiener Theorem on SU(1,1).

We assume the Paley-Wiener theorem for $n=\varepsilon$. Then we shall give a proof for arbitrary n by reducing it to the case $n=\varepsilon$.

Let PW be the subspace of $L^2_n(R\times T) \oplus A^2_n(D)$ defined by

$$PW = \{\gamma=(\alpha(\lambda,\zeta), \beta(z)) \in L^2(R\times T, \mu_\varepsilon(\lambda)d\lambda d\zeta) \oplus A^2_n(D);$$

(1) $\alpha(\lambda,\zeta)$ is an antiholomorphic function of uniform exponential type,

(2) $A_n(\alpha)(\lambda,x) = A_n(\alpha)(-\lambda,x) \quad ((\lambda,x)\in R\times G)$,

(3) $\alpha(-(m-½)i,\zeta) = \lambda_{n-m}^{-1}\ \beta_m(\zeta)e_{\varepsilon-m}(\zeta)\ (\zeta\in T)$, where $\beta(z)=(\beta_m(z);\ m\in I_n)\ \}$.

Then we can easily deduce that the Fourier transforms f^ of f in $C^\infty_{c,n}(G)$ satisfy (1) and (2). Furthermore, the following theorem guarantees that they also satisfy (3). In fact, this theorem implies that the realization of the discrete series T_m $(m\in I_n)$ on the weighted Bergman space is nothing but the one on the image of the Szegö operators $S_{\varepsilon,m,m+1}$.

Theorem 3. Let $\varepsilon \in \{0, ½\}$, $m \in Z_\varepsilon$ and $m \geqq -½$.

If $f(\zeta) = \sum_{p \in Z_\varepsilon} a_p \zeta^p$ be a function in $L^2(K, \sigma_\varepsilon)$ satisfying $|a_p| = 0$ for $|p| \leqq m$, then

$$S_{\varepsilon,m,m+1}(f)(x)$$

$$= (1-|w|^2)^{m+½} e^{i(m+½)(\theta+\theta')} S_{½,-½,½}(fe^{-i(m+½)\theta})$$

$$= (1-|w|^2)^{m+1} e^{i(m+1)(\theta+\theta')} (I_\varepsilon^{-1}f)_+(w) w^{-(m+1)},$$

where $x = k_\theta a_t k_{\theta'} \in G$ and $w = x \cdot 0 \in D$.

On the other hand, the reverse < for each $\gamma \in PW$ $\gamma^\vee$ belongs to $C^\infty_{c,n}(G)$ > can be shown as follows.

① $f \to {}^\circ f$ is surjective of $C^\infty_{c,n}(G)$ onto ${}^\circ C_n(G)$.

② We choose $g \in C^\infty_{c,n}(G)$ such that ${}^\circ g = \beta$.

③ Put $h = \gamma^\vee - g$. Then $h^\wedge \in PW$ and $h^\wedge(z) \equiv 0$.

④ (3) implies $h^\wedge(\lambda, \zeta)$ has zero at the pole of $P_n(\lambda)^{-1}$.

⑤ $h^\wedge(\lambda, \zeta) P_n(\lambda)^{-1}$ is an antiholomorphic function of uniform exponential type.

⑥ The Paley-Wiener theorem for $n=\varepsilon$ deduces $h={}^{P}h \in C_{c,n}^{\infty}(G)$.

⑦ $\gamma^{\vee} = h + g \in C_{c,n}^{\infty}(G)$.

Therefore, by reducing the proof to the case of $n=\varepsilon$, we can obtain the following

Theorem 4. (Paley-Wiener Theorem on SU(1,1))
The Fourier transform $f \to f^{\wedge}$ is a bijection of $C_{c,n}^{\infty}(G)$ onto PW.

By the same way we can give a characterization of the L^{p} Schwartz space on G.