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Harmonic Analysis on Semisimple Symmetric Spaces

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1. SYMMETRIC SPACES

Definition 1.1. A manifold M where a Lie group G acts smoothly and transitively is called *a homogeneous space* of G.

For a point $x \in M$ put $G^x = \{g \in G;\ gx = x\}$. Then

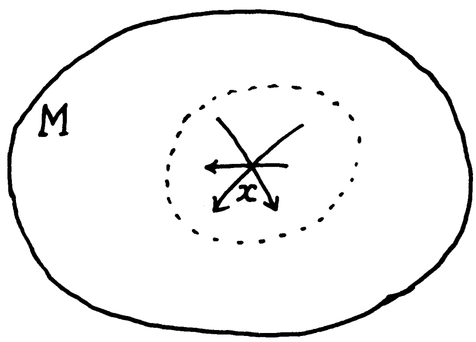
$$M \simeq G/G^x.$$

In general, for a close subgroup H of G, G/H has a unique structure of a C^ω-manifold where the left action of G is smooth. Hence we identify G/G^x with M and call them a homogeneous space of G.

Definition 1.2. A homogeneous space $M = G/H$ is called *symmetric* if there exists an automorphism σ of G so that $\sigma^2 = id$ and that $H = G^\sigma$ in neighborhoods of the identity of the Lie groups.

$$\sigma \in \mathrm{Aut}(G),\ \sigma^2 = id \quad \text{and} \quad H:\ \text{open} \subset G^\sigma = \{g \in G;\ \sigma(g) = g\}$$

Definition 1.2′. Let M be a manifold with an affine connection ∇. For a point $x \in M$, let S_x be a local point symmetry around x which is defined by geodesic flows through x. If S_x can be extended for a global isomorphism of (M, ∇) for any $x \in M$, then M is called a symmetric space.



This article is prepared for the workshop celebrating I. M. Gelfand for a Kyoto Prize winner.

Definition 1.3. A symmetric space G/H is called *semisimple symmetric space* if G is a semisimple Lie group. (G is locally isomorphic to a closed subgroup G' of $GL(n,\mathbf{R})$ so that ${}^tG' = G'$ and that the center of G' is finite.)

A symmetric space G/H is *a Riemannian symmetric space* if G/H has a G-invariant Riemannian metric.

A semisimple symmetric space has a G-invariant pseudo-Riemannian metric.

Example 1. $M = G'$: A Lie group.

$$\begin{array}{ccccc} G(=G'\times G')\times M & \rightarrow & M & & M \simeq G/H \\ \Psi & & \Psi & & H = G^\sigma = \Delta G \\ ((g_1,g_2),x) & \mapsto & g_1 x g_2^{-1} & & \sigma(g_1,g_2) = (g_2,g_1) \end{array}$$

Example 2. G is a semisimple Lie subgroup of $GL(n,\mathbf{R})$ with ${}^tG = G$ and $H = G \cap SO(n)$.

G/H: A Riemannian symmetric space of the non-compact type.

Example 3.
$$SL(m+n,\mathbf{R})/SO(m,n)$$
$$SL(n,\mathbf{C})/SL(n,\mathbf{R})$$
$$SL(2n,\mathbf{R})/U(GL(n,\mathbf{C}))$$
$$SL(2n,\mathbf{R})/Sp(n,\mathbf{R})$$
$$SL(m+n,\mathbf{R})/S(GL(m,\mathbf{R})\times GL(n,\mathbf{R}))$$
$$SO_o(m,n)/SO_o(p,q)\times SO_o(m-p,n-q), \text{ etc.}$$

A local classification of semisimple symmetric spaces is given by Berger [Ann. Sci. École Norm. Sup.,**74**(1957), 85–177].

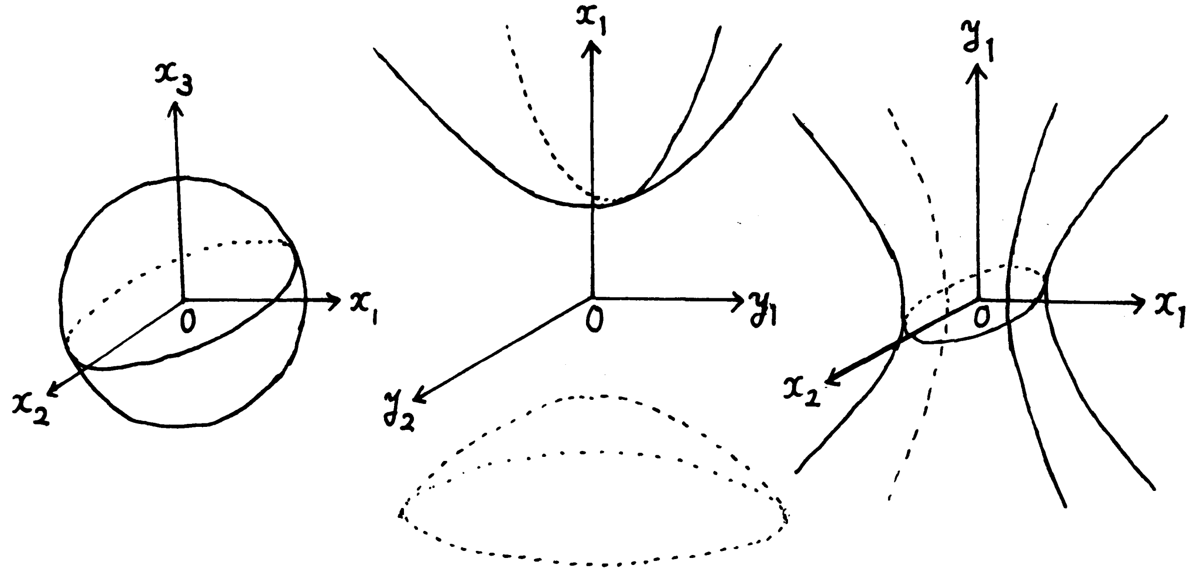
Example 4. $H_{m,n} = SO_o(m,n)/SO_o(m-1,n)$

$H_{m,n}$: The connected component ($\ni (1,0,\ldots,0)$) of
$$x_1^2 + \cdots + x_m^2 - y_1^2 - \cdots - y_n^2 = 1 \subset \mathbf{R}^{m+n}$$

$H_{m,0}$: $(m-1)$-dimensional spherical surface

A Riemannian symmetric space of the compact type

$H_{1,n}$: A Riemannian symmetric space of the non-compact type



$H_{3,0}$ $\qquad$ $H_{1,2}$ $\qquad$ $H_{2,1}$

2. Harmonic Analysis

There is a notion of a direct product of symmetric spaces, subsymmetric spaces, quotient symmetric spaces, irreducible symmetric spaces etc. Any irreducible symmetric space is a semisimple symmetric space and so the semisimple symmetric space is fundamental.

Hereafter we assume that $M = G/H$ is a semisimple symmetric space.

Problem 1. Find an explicit decomposition of $L^2(G/H)$ into irreducible unitary representations of G.

The ring $D(M)$ of G-invariant differential operators on M is isomorphic to a polynomial ring

$$D(M) \simeq \mathbf{C}[\Delta_1, \ldots, \Delta_\ell].$$

Here ℓ is a rank of M and it is equal to the maximal dimension of totally flat submanifolds of M. Δ_i can be assumed self-adjoint operators in $L^2(M)$.

Problem 2. Find a simultaneous spectral decomposition of $D(M)$.

Problem 1 is almost equivalent to Problem 2.

3. Smooth Imbeddings and Flensted-Jensen's Duality

K: A σ-stable maximal compact subgroup of G modulo center of G.

$M^r = G^d/K^d$: A Riemannian symmetric space of the non-compact type satisfying

$$\mathrm{Lie}(G) \otimes \mathbf{C} = \mathrm{Lie}(G^d) \otimes \mathbf{C}, \quad \mathrm{Lie}(H) \otimes \mathbf{C} = \mathrm{Lie}(K^d) \otimes \mathbf{C}$$

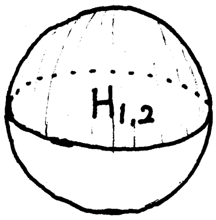
$G_{\mathbf{C}}$: A Lie group with $\mathrm{Lie}(G_{\mathbf{C}}) = \mathrm{Lie}(G) \otimes \mathbf{C}$.

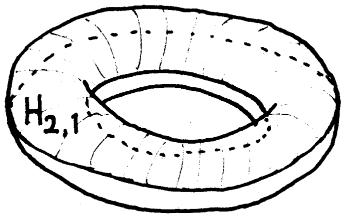
$K_{\mathbf{C}}$: The analytic subgroup of $G_{\mathbf{C}}$ with the Lie algebra $\mathrm{Lie}(K) \otimes \mathbf{C}$.

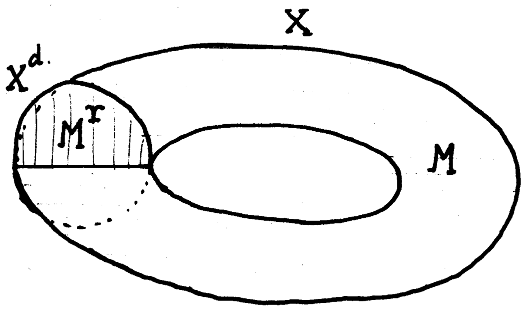
Definition 3.1. M^r is called a non-compact Riemannian form of M.

Examples. $M \longleftrightarrow M^r$: $\quad SL(n, \mathbf{R}) \longleftrightarrow SL(n, \mathbf{C})/U(n)$

$$SL(m+n, \mathbf{R})/SO(m,n) \longleftrightarrow SL(m+n, \mathbf{R})/SO(m+n)$$







$$\begin{array}{cccc}
X = G/P_\sigma & X^d = G^d/P^d & V(\Lambda) & V(\lambda)^r \\
\downarrow & \downarrow & \uparrow \beta_\Lambda & \uparrow \beta_\lambda^r \ \wr \ \text{Helgason's conj.} \\
\widetilde{M} & \widetilde{M^r} & & \\
\uparrow & \uparrow & & \\
M & M^r & E(\lambda) & E(\lambda)^r \\
& & \cup & \cup \\
& & E(\lambda)_{K_{\mathbf{C}}} & \overset{\sim}{\longleftrightarrow} \ E(\lambda)^r_{K_{\mathbf{C}}}
\end{array}$$

(Flensted-Jensen isomorphism)

$A \simeq \mathbf{R}_{\times}^{\ell'}$: A maximal split torus of G with $\sigma(a) = a^{-1}$ ($\forall a \in A$).

$\widetilde{M}$ is a compact G-manifold and there is a smooth G-equivariant map

$$\begin{array}{ccc}
G \times \{0\} & \twoheadrightarrow & X \simeq G/P_\sigma \\
\downarrow & & \downarrow \\
G \times \mathbf{R}^{\ell'} & \rightarrow & \widetilde{M} \\
\uparrow & & \uparrow \\
G \times A & \twoheadrightarrow & M.
\end{array}$$

Here P_σ is a parabolic subgroup of G whose Levi part equals the centralizer of A in G.

λ is the parameter for the algebra homomorphisms χ_λ of $D(\widetilde{M}) \simeq D(M)$ to $\mathbf{C}$. Then the system

$$\mathcal{M}_\lambda : \ \Delta u = \chi_\lambda(\Delta)u \quad \forall \Delta \in D(\widetilde{M}),$$

has regular singularities along the boundaries of M in $\widetilde{M}$ in the sense of [Kashiwara-Oshima, Ann. of Math., **106**(1977), 145–200]. Denoting $E(\lambda)$ the hyperfunction solution space of the system on M, we can define a boundary value map β_λ of $V(\lambda)$ into the space $V(\Lambda)$ of the hyperfunction sections of a principal series for G/H. The principal series are the representations of G induced from finite dimensional irreducible representations of P_σ with $H \cap P_\sigma$-fixed vectors.

4. Discrete Series for G/H

Definition 4.1. An irreducible unitary representation of G (or its equivalence class) is called a representation of G belonging to a generalized *discrete series* for G/H if it is realized in the function space of G whose elements are square integrable modulo center of G.

Theorem 4.2 (Flensted-Jensen, Matsuki-Oshima). The generalized discrete series for G/H is non-empty if and only if

$$\operatorname{rank} G/H = \operatorname{rank} K/K \cap H.$$

This condition is equivalent to the following:

There exists a totally flat compact submanifold of M whose dimension equals $\operatorname{rank} M$.

Theorem 4.3 (Matsuki-Oshima). Under this condition the Harish-Chandra modules corresponding to the discrete series for G/H are isomorphic to the relative cohomology groups

$$H^k_{[V_i]}(X^d_{\mathbf{C}}; L_\lambda)$$

in the Zariski topology. Here $X^d_{\mathbf{C}}$ is the generalized flag manifold $G_{\mathbf{C}}/P^d_{\mathbf{C}}$ with the complexification $P^d_{\mathbf{C}}$ of a minimal parabolic subgroup of G^d, L_λ is a holomorphic line bundle over $X_{\mathbf{C}}$, V_i are the compact $K_{\mathbf{C}}$ orbits in $X^d_{\mathbf{C}}$ with representatives in G^d and k is the codimension of the orbits.

5. C-FUNCTIONS AND PLANCHEREL MEASURES

Definition 5.1. Normalized H-invariant distribution sections P_Λ of the principal series for G/H are called Poisson kernels for G/H. The Poisson transformation $\mathcal{P}_\Lambda$ is defined by

$$\begin{array}{ccc} V(\Lambda) & \to & E(\lambda) \\ \cup\!\!\!\!\!\!\;\;\shortmid & & \cup\!\!\!\!\!\!\;\;\shortmid \\ v(g) & \mapsto & (\mathcal{P}_\Lambda v)(x) = \int_K v(k)P_\lambda(g^{-1}k)dk. \end{array}$$

Then the constant $C(\Lambda) = \beta_\Lambda \circ \mathcal{P}_\Lambda$ is called the c-function for M.

Theorem 5.2 (Oshima-Sekiguchi). There is an explicit formula of c-function in terms of the Γ-functions.

The essential part of the proof of this theorem is to show that $C(\Lambda)$ satisfies the same difference equations of the c-function for M^r with respect to the lattice generated by the weights of finite dimensional representations of G with H-fixed vectors. Note that the c-function for M^r is calculated by Gindikin-Karpelevič.

Theorem 5.3. The integral transformation with the kernel function P_Λ of $L^2(M)$ defines a projection operator of $L^2(M)$ onto unitary principal series for M. Combining this with the Poisson transform, the Plancherel measures for unitary principal series is given by $C(\lambda)^{-1}C(-\lambda)^{-1}$ under a certain normalization of Haar measures of G and H.

Theorem 5.4. For a representative A_i of an H-conjugate class of split abelian subgroups of G with $\sigma(a) = a^{-1}$ ($\forall a \in A_i$), let P_i denote the parabolic subgroup of G whose Levi part L_i equals the centralizer of A_i. We can define unitary representations $V_i(\Lambda)$ of G induced from the irreducible representations of P_i corresponding to generalized discrete series for the symmetric space $L_i/L_i \cap H$. Then we can define a projection operators, Poisson kernels and c-functions for $V_i(\Lambda)$ in a similar way and the corresponding Plancherel measures are expressed by the c-functions, which are essentially parts of the c-function in Theorem 5.2. Using these representations, we can decompose $L^2(M)$ into irreducible unitary representations of G.

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