A characterization of nilpotent varieties of complex semisimple Lie algebras

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Introduction

A normal complex algebraic variety $X$ is called a symplectic variety (cf. [Be]) if its regular locus $X_{\text{reg}}$ admits a holomorphic symplectic 2-form $\omega$ such that it extends to a holomorphic 2-form on a resolution $f : \tilde{X} \to X$.

Affine symplectic varieties are constructed in various ways such as nilpotent orbit closures of a semisimple complex Lie algebra (cf. [CM]), Slodowy slices to nilpotent orbits (cf. [Sl]) or symplectic reductions of holomorphic symplectic manifolds with Hamiltonian actions. Usually these examples show up with $\mathbb{C}^*$-actions.

In this lecture we shall characterize the nilpotent variety of a complex semisimple Lie algebra among affine symplectic varieties from a viewpoint of algebraic geometry.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $N$ be the nilpotent variety of $\mathfrak{g}$. It is well known that $N$ is an affine normal variety and its regular locus admits a holomorphic symplectic 2-form $\omega_{KK}$ called the Kostant-Kirillov 2-form. Then $(N, \omega_{KK})$ is an affine symplectic variety in our sense. Moreover, the scalar multiplication determines a $\mathbb{C}^*$-action on $\mathfrak{g}$ and it induces a $\mathbb{C}^*$-action also on $N$. The Kostant-Kirillov 2-form $\omega_{KK}$ has weight 1 with respect to this $\mathbb{C}^*$-action. The adjoint group $G$ acts on $\mathfrak{g}$ and let $\mathfrak{g}/G$ be the GIT quotient of the $G$-action. Namely $\mathfrak{g} := \text{Spec} \mathbb{C}[\mathfrak{g}]^G$, where $\mathbb{C}[\mathfrak{g}]^G$ is the $G$-invariant ring of the coordinate ring $\mathbb{C}[\mathfrak{g}]$ of $\mathfrak{g}$. By a theorem of Chevalley, $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to a polynomial ring $\mathbb{C}[f_1, ..., f_r]$ generated by algebraically independent $G$-invariant homogeneous polynomials $f_i$. Here $r$ coincides with the rank of $\mathfrak{g}$. Let $\chi : \mathfrak{g} \to \mathfrak{g}/G = \mathbb{C}^r$ be the adjoint quotient map. Then $N = \chi^{-1}(0)$. In particular, $N$ is a complete intersection of $r$ homogeneous polynomials in the affine space $\mathfrak{g}$. 

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Our main theorem asserts that the converse holds true. More precisely, let $(X, \omega)$ be a $2n$-dimensional affine symplectic variety embedded in the affine space $\mathbb{C}^{2n+r}$ as a complete intersection of $r$ homogeneous polynomials $f_i(z_1, \ldots, z_{2n+r}) = 0$ (1 $\leq i \leq r$). The affine space $\mathbb{C}^{2n+r}$ has a standard $\mathbb{C}^*$-action with $wt(z_i) = 1$ for all $i$. It induces a $\mathbb{C}^*$-action on $X$. We assume that the symplectic form $\omega$ is homogeneous with respect to this $\mathbb{C}^*$-action. Namely, for some integer $l$, we have $t^*\omega = t^l \cdot \omega$ where $t \in \mathbb{C}^*$. The integer $l$ is called the weight of $\omega$ and is denoted by $wt(\omega)$. When $X$ is smooth, $(X, \omega)$ is isomorphic to $(\mathbb{C}^{2n}, \omega_0)$, where $\omega_0$ is the standard symplectic 2-form $\Sigma dz_{2i-1} \wedge d\bar{z}_{2i}$. In the remainder we restrict ourselves to the case when $X$ is singular. Then we have:

**Main Theorem.** There is a $\mathbb{C}^*$-equivariant isomorphism $(X, \omega) \cong (N, \omega_{KK})$ of symplectic varieties. Here $N$ is the nilpotent variety of a complex semisimple Lie algebra $\mathfrak{g}$ and $\omega_{KK}$ is the Kostant-Kirillov 2-form.

The proof consists of two steps. At first we prove that $X$ coincides with a nilpotent orbit closure $\bar{O}$ of a semisimple complex Lie algebra $\mathfrak{g}$ (Theorem 2). Theorem 2 actually shows that $X$ is the closure of a Richardson orbit $O$ and $\bar{O}$ has a crepant resolution. We next prove in 6 that such a nilpotent orbit closure $\bar{O}$ must be the nilpotent variety $N$ if it has complete intersection singularities.

A symplectic variety tends to have a large embedded codimension. The main theorem shows that the $A_1$ surface singularity is a unique homogeneous symplectic hypersurface. As is studied in [LNSV] we have some examples of quasihomogeneous symplectic hypersurfaces in higher dimensions.

The results of this note are concerned with symplectic varieties. However the proof of Theorem 2 is based on contact geometry. In particular, a structure theorem [KPSW] on contact projective manifolds plays a crucial role.

The results of this note have already been published in [Na].

1. Let $X$ be a homogeneous symplectic variety of complete intersection defined in Introduction.

When $X$ is smooth, the polynomials $f_i$ are all linear forms; hence we may assume that $r = 0$ and $X = \mathbb{C}^{2n}$. We can write $\omega^n := \omega \wedge \ldots \wedge \omega = g \cdot dz_1 \wedge \ldots \wedge d\bar{z}_{2n}$ with a nowhere vanishing homogeneous polynomial $g$. Since such a polynomial $g$ must be a constant, we have $wt(\omega) = 2$. Now $\omega$ has a form $\Sigma a_{ij}dz_i \wedge d\bar{z}_j$ with some constants $a_{ij}$. Then $\omega$ becomes the standard
symplectic 2-form $\Sigma_{1 \leq i \leq n} dz_{2i-1} \wedge dz_{2i}$ after a suitable linear transformation of $\mathbb{C}^{2n}$.

From now on we consider the case when $X$ is singular. Without loss of generality we may assume that $\deg(f_i) \geq 2$ for all $i$. In the remainder we put $a_i := \deg(f_i)$. By the adjunction formula (or the residue formula) we have

$$\omega^n = c \cdot \text{Res}_X(dz_1 \wedge ... \wedge dz_{2n+r}/(f_1, ..., f_r))$$

with a nonzero constant $c$; hence

$$\text{wt}(\omega^n) = 2n + r - \Sigma a_i.$$ 

Since $\text{wt}(\omega^n) = n \cdot \text{wt}(\omega)$ and $\text{wt}(\omega) > 0$ (cf. [LNSV], Lemma 2.2), we have $\Sigma a_i = n + r$ and $\text{wt}(\omega) = 1$.

We next consider a resolution of the singular variety $X$. Since $X$ is a normal Gorenstein singularity, its canonical divisor $K_X$ is a Cartier divisor. A resolution $\pi : Y \to X$ is called crepant if $K_Y = \pi^* K_X$. A general symplectic variety does not have a crepant resolution, but our $X$ has because it is of complete intersection:

**Theorem 1.** $X$ has a $\mathbb{C}^*$-equivariant crepant resolution $\pi : Y \to X$.

**Proof.** Let us take a resolution $g : W \to X$ and apply the minimal model program to $g$ ([BCHM]). We then finally get a $\mathbb{Q}$-factorial terminalisation $\pi : Y \to X$ of $X$. Namely $Y$ has only $\mathbb{Q}$-factorial terminal singularities and $K_Y = \pi^* K_X$. We shall prove that $Y$ is actually smooth.

The pullback $\pi^* \omega$ defines a symplectic structure on the regular part of $Y$. Let $f : Z \to Y$ be a resolution of $Y$. By the assumption $(\pi \circ f)^* \omega$ extends to a holomorphic 2-form on $Z$; hence $Y$ is a symplectic variety. Then $\text{Sing}(Y)$ has even codimension by Kaledin [Ka]. On the other hand, since $Y$ has only terminal singularities, $\text{Codim}_{Y} \text{Sing}(Y) \geq 3$. Hence we have $\text{Codim}_{Y} \text{Sing}(Y) \geq 4$. Moreover the $\mathbb{C}^*$-action on $Y$ extends to a $\mathbb{C}^*$-action on $Y$ (cf. [Na 1, Proposition A.7]). Note here that a symplectic variety has a natural Poisson structure and one can consider its Poisson deformation (cf. [Na 2]). Take a Poisson deformation $Y_t$ of $Y$. Then the birational map $\pi : Y \to X$ also deforms to a birational map $\pi_t : Y_t \to X_t$, where $X_t$ is a Poisson deformation of $X$. If we take the Poisson deformation $Y_t$ general enough, then $\pi_t$ is an isomorphism (cf. [Na 2, Theorem 5.5]). In particular, $Y_t = X_t$. Since $X$ has only complete intersection singularities, so does $Y_t$. On the other hand, $\text{Codim}_{Y_t} \text{Sing}(Y_t) \geq 4$. By a result of Beauville [Be, Proposition
a symplectic singularity is a complete intersection singularity only if its singular locus has codimension $\leq 3$. Therefore, $Y_t$ must be smooth. Since $Y$ has only $\mathbb{Q}$-factorial terminal singularities, any Poisson deformation of $Y$ is locally trivial as a flat deformation by Proposition A.9 and Theorem 17 of [Na 1]. This means that $Y$ is smooth. Q.E.D.

Here let us recall the notion of a contact structure. A complex manifold $M$ of dimension $2n - 1$ has a contact structure if there is an exact sequence of vector bundles on $M$:

$$0 \to D \to \Theta_M \xrightarrow{\eta} L \to 0,$$

where $\text{rank}(D) = 2n - 2$, $L$ is a line bundle, and $D \times D \to L$, $(x, y) \mapsto \eta([x, y])$ is a non-degenerate pairing. Notice that $\eta$ can be regarded as a section of $\Omega^1_M \otimes L$. We call this twisted 1-form a contact form and call the line bundle $L$ a contact line bundle.

A contact structure is naturally introduced in the following situation. Assume that $M$ is a complex manifold and $L$ is a line bundle on $M$. We put $(L^{-1})^\times := L^{-1} - \{0\}$ and let $p : (L^{-1})^\times \to M$ be the projection map. As $(L^{-1})^\times$ is a $\mathbb{C}^*$-bundle, there is a natural $\mathbb{C}^*$-action on $(L^{-1})^\times$. The $\mathbb{C}^*$-action determines a vector field $\zeta$ on $(L^{-1})^\times$. Assume that $(L^{-1})^\times$ admits a holomorphic symplectic 2-form $\omega$ of weight 1 with respect to the $\mathbb{C}^*$-action. Then the 1-form $i_\zeta \omega$ on $(L^{-1})^\times$ has weight 1 and we can write $i_\zeta \omega = p^* \eta$ for $\eta \in \Gamma(M, \Omega^1_M \otimes L)$. Then this $\eta$ determines a contact structure on $M$.

We can apply this construction to the projectivisation $\mathbf{P}(X) := X - \{0\}/\mathbb{C}^*$ of $X$. The $\mathbb{C}^*$-bundle $O_{\mathbf{P}(X)}(-1)^\times \to \mathbf{P}(X)$ induces a $\mathbb{C}^*$-bundle $O_{\mathbf{P}(X)_{\text{reg}}}(-1)^\times \to \mathbf{P}(X)_{\text{reg}}$. As $X_{\text{reg}}$ is identified with $O_{\mathbf{P}(X)_{\text{reg}}}(-1)^\times$, there is a holomorphic symplectic 2-form $\omega$ of weight 1 on it. Then it determines a contact structure on $\mathbf{P}(X)_{\text{reg}}$ (cf. [LeB], [Na 3, Section 4]). More precisely there is an exact sequence of vector bundles on $\mathbf{P}(X)_{\text{reg}}$:

$$0 \to D \to \Theta_{\mathbf{P}(X)_{\text{reg}}} \xrightarrow{\eta} O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{\text{reg}}} \to 0,$$

where $\eta$ is a contact 1-form.

2. We first claim that $\mathbf{P}(X)$ also has a crepant resolution $^1$. Let $L$ be a $\pi$-ample line bundle on $Y$. If necessary, replacing $L$ by its suitable multiple, we may assume that $L$ has a $\mathbb{C}^*$-linearisation (cf. [CG] Theorem 5.1.9). We

$^1$This is a crucial conclusion obtained from the assumption $wt(z_i)$ are all 1.
put $A_m := \Gamma(Y, L^{\otimes m})$ for each $m \geq 0$. Note that each $A_m$ has a grading determined by the $\mathbb{C}^*$-action. In particular, $A_0$ is the coordinate ring of $X$ and $\mathbf{P}(X) = \text{Proj}(A_0)$. Since $A_m$ are graded $A_0$-modules, we can consider the associated coherent sheaves $A_m$ on $\mathbf{P}(X)$. Define $Z := \mathbf{P}(\mathbf{P}(X) \oplus A_m)$. Then $Z$ can be identified with $Y - \pi^{-1}(0)/\mathbb{C}^*$ and the projective morphism $\tilde{\pi} : Z \to \mathbf{P}(X)$ can be identified with the natural map $Y - \pi^{-1}(0)/\mathbb{C}^* \to X - \{0\}/\mathbb{C}^*$ induced by the $\mathbb{C}^*$-equivariant resolution $\pi : Y \to X$. In particular, $\tilde{\pi}$ is a birational map. Look at the commutative diagram

\[
\begin{array}{ccc}
Y - \pi^{-1}(0) & \longrightarrow & Y - \pi^{-1}(0)/\mathbb{C}^* \\
\downarrow & & \downarrow \\
X - \{0\} & \longrightarrow & X - \{0\}/\mathbb{C}^*.
\end{array}
\]

(1)

Pick a point $x := (z_1(x), ..., z_{2n+r}(x)) \in X - \{0\}$. We have $z_i(x) \neq 0$ for some $i$. Define $U_x := X \cap \{(z_1, ..., z_{2n+r}) \in \mathbb{C}^{2n+r}; z_i = z_i(x)\}$. Then $U_x$ is isomorphically mapped onto a Zariski open subset of $\mathbf{P}(X)$ by the map $X - \{0\} \to \mathbf{P}(X)$. The map

$\sigma_x : \mathbb{C}^* \times U_x \to X - \{0\}$

sending $(t, x') \in \mathbb{C}^* \times U_x$ to $t \cdot x' \in X - \{0\}$ is an open immersion. We put $V_x := \pi^{-1}(U_x)$. Choose a point $y' \in V_x$ and put $x' := \pi(y')$. Denote by $O_{x'}$ (resp. $O_{y'}$) the $\mathbb{C}^*$-orbit of $x'$ (resp. $y'$).

Since $O_{x'}$ and $O_{y'}$ are both $\mathbb{C}^*$ orbits, there are natural surjections $\gamma_{x'} : \mathbb{C}^* \to O_{x'}$ ($t \mapsto t \cdot x'$) and $\gamma_{y'} : \mathbb{C}^* \to O_{y'}$ ($t \mapsto t \cdot y'$). Moreover $\gamma_{y'}$ factorizes $\gamma_{x'}$:

$\mathbb{C}^* \xrightarrow{\gamma_{y'}} O_{y'} \to O_{x'}$.

Since $wt(z_i) = 1$ for all $i$, we see that $\gamma_{x'}$ is an isomorphism; hence $\gamma_{y'}$ is also an isomorphism and $O_{y'} \cong O_{x'}$.

Let $T_{y'}V_x$ (resp. $T_{y'}O_{y'}$) be the tangent space of $V_x$ (resp. $O_{y'}$) at $y'$. Then one has

$T_{y'}V_x \cap T_{y'}O_{y'} = \{0\}$.

In fact, the isomorphism $O_{y'} \to O_{x'}$ induces an isomorphism of the tangent spaces $T_{y'}O_{y'} \to T_{x'}O_{x'}$. This isomorphism induces an injection $T_{y'}V_x \cap T_{y'}O_{y'} \to T_{x'}U_x \cap T_{x'}O_{x'}$. Since $T_{x'}U_x \cap T_{x'}O_{x'} = \{0\}$ by the construction of $U_x$, we see that $T_{y'}V_x \cap T_{y'}O_{y'} = \{0\}$. 
Let us consider the map

\[ \sigma_{V_x} : C^* \times V_x \rightarrow Y - \pi^{-1}(0). \]

This map induces a map of tangent spaces

\[ T_{(t, y')} (C^* \times V_x) \rightarrow T_{t \cdot y'} Y \]

for \((t, y') \in C^* \times V_x\).

We claim that \(V_x\) is smooth at \(y'\) and this map of tangent spaces is an isomorphism. We first show the injectivity. We identify \(C^* \cdot t \rightarrow t\)

\(C^*\) to an element of \(C^* \cdot t\), and \(\alpha \rightarrow \gamma_{y_1} : C^* \rightarrow O_{y_1}\). Moreover this implies that the map is an isomorphism.

By the assumption \(\sigma_{V_x}\) is an isomorphism, we see that \(\sigma_{V_x} : C^* \times V_x \rightarrow C^* \times V_x\) is also a crepant resolution. This means that \(V_x \rightarrow U_x\) is also a crepant resolution.
Therefore we get a crepant resolution $\bar{\pi} : Z \to \mathbb{P}(X)$ of $\mathbb{P}(X)$.

3. We next claim that $Z$ is a contact projective manifold with the contact line bundle $\bar{\pi}^*O_{\mathbb{P}(X)}(1)$.

For simplicity we write $L$ for $O_{\mathbb{P}(X)}(1)|_{\mathbb{P}(X)_{\text{reg}}}$. The contact structure on $\mathbb{P}(X)_{\text{reg}}$ is expressed as a twisted 1-form $\eta \in \Gamma(\mathbb{P}(X)_{\text{reg}}, \Omega^1_{\mathbb{P}(X)_{\text{reg}}} \otimes L)$ such that $\eta \wedge (du)^{n-1} \in O_{\mathbb{P}(X)_{\text{reg}}}$ is nowhere-vanishing. In our case $L$ extends to the line bundle $O_{\mathbb{P}(X)}(1)$ on $\mathbb{P}(X)$. Let $i : \mathbb{P}(X)_{\text{reg}} \to \mathbb{P}(X)$ be the natural inclusion map. Since $\mathbb{P}(X)$ has only canonical singularities, we have $\bar{\eta}, \Omega^1_Z \cong i_*\Omega^1_{\mathbb{P}(X)_{\text{reg}}} (\text{[GKK]})$. Hence the pull-back $\bar{\eta}$ is a section of $\Omega^1_Z \otimes \bar{\pi}^*O_{\mathbb{P}(X)}(1)$. Moreover, since $\bar{\pi}$ is a crepant resolution, $\bar{\pi}^*\eta \wedge (d\bar{\pi}^*\eta)^{n-1}$ is nowhere-vanishing.

Therefore we get a contact structure of $Z$ with the contact line bundle $\bar{\pi}^*O_{\mathbb{P}(X)}(1)$.

4. When $n = 1$ we already know that $r = 1$ and $f = z_1^2 + z_2^2 + z_3^2$ after a suitable change of coordinates (cf. [LNSV], 3.1). Note that $Z = \mathbb{P}(X) = \mathbb{P}^1$ in this case. We assume that $n \geq 2$. Then $\text{Codim}_X\text{Sing}(X) = 2$ by [Be, Proposition 1.4]. Hence $\mathbb{P}(X)$ actually has singularities and $b_2(Z) \geq 2$. Note that $K_Z$ is not nef because $K_Z = \bar{\pi}^*O_X(-n)$. Now we apply the following structure theorem of Kebekus, Peternell, Sommese and Wisniewski [KPSW].

**Theorem.** Let $Z$ be a contact projective manifold with a contact line bundle $L$. Assume that $b_2(Z) \geq 2$ and $K_Z$ is not nef. Then $Z$ is isomorphic to the projectivised cotangent bundle $\mathbb{P}(\Theta_M)^2$ of a projective manifold $M$ of dimension $n$; moreover, $L \cong O_{\mathbb{P}(\Theta_M)}(1)$.

Since the contact line bundle is $\bar{\pi}^*O_{\mathbb{P}(X)}(1)$ in our case, we have $\bar{\pi}^*O_{\mathbb{P}(X)}(1) \cong O_{\mathbb{P}(\Theta_M)}(1)$.

Let $\eta_0$ be the canonical contact structure on $\mathbb{P}(\Theta_M)$ induced by the canonical symplectic form on $T^*M$. Note here that an automorphism $\varphi$ of the vector bundle $\Theta_M$ induces an automorphism of $Z := \mathbb{P}(\Theta_M)$, which is denoted by the same notation $\varphi$. Then $\Omega^1_Z$ and $O_{\mathbb{P}(\Theta_M)}(1)$ are both $\text{Aut}(\Theta_M)$-linearized. Then our contact form $\eta$ can be written as $\eta = \varphi^*\eta_0$ for some $\varphi \in \text{Aut}(\Theta_M)$ (cf. [KPSW], Proposition 2.14). We may assume that $\eta = \eta_0$ by composing $\varphi$ with the initial identification $Z \cong \mathbb{P}(\Theta_M)$.

The embedding $X \to \mathbb{C}^{2n+r}$ induces an embedding $\mathbb{P}(X) \to \mathbb{P}^{2n+r-1}$.  

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In this note we employ Grothendieck’s notation for a projective space bundle. Namely $\mathbb{P}(\Theta_M) = T^*M - (0 - \text{section})/\mathbb{C}^*$. 

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Since \( H^0(\mathbb{P}^{2n+r-1}, O_{\mathbb{P}^{2n+r-1}}(1)) \cong H^0(\mathbb{P}(X), O_{\mathbb{P}(X)}(1)) \), the morphism \( \bar{\pi} \) coincides with the one defined by the complete linear system \( |O_{\mathbb{P}(\Theta_M)}(1)| \).

**Lemma.** \( \chi(\mathbb{P}(X), O_{\mathbb{P}(X)}) = 1 \).

**Proof.** We first claim that if \( W \subset \mathbb{P}^n \) is a complete intersection of type \((d_1, \ldots, d_k)\), then \( \chi(W, O_W(-i)) = 0 \) for all \( i > 0 \) with \( d_1 + \ldots + d_k + i < m + 1 \). We prove this by the induction on \( k \). Assume that this is true for \( k - 1 \). Let us take the complete intersection \( W' \) of type \((d_1, \ldots, d_{k-1})\) such that \( W \) is an element of \( |O_{W'}(d_k)| \). By the exact sequence

\[
0 \to O_{W'}(-i - d_k) \to O_{W'}(-i) \to O_{W'}(-i) \to 0
\]

we have \( \chi(O_W(-i)) = \chi(O_{W'}(-i)) - \chi(O_{W'}(-i - d_k)) \). Assume that \( d_1 + \ldots + d_k + i < m + 1 \). Then we have \( d_1 + \ldots + d_{k-1} + (i + d_k) < m + 1 \) and \( d_1 + \ldots + d_{k-1} + i < m + 1 \). By the induction assumption \( \chi(O_{W'}(-i - d_k)) = \chi(O_{W'}(-i)) = 0 \); hence \( \chi(O_W(-i)) = 0 \).

We next claim that \( \chi(W', O_{W'}) = 1 \) if \( d_1 + \ldots + d_k < m + 1 \). This is also proved by the induction on \( k \). We take the same \( W' \) as above. Then \( \chi(O_W) = \chi(O_{W'}) - \chi(O_{W'}(-d_k)) \). By the induction assumption \( \chi(O_{W'}) = 1 \). By the previous claim we have \( \chi(O_{W'}(-d_k)) = 0 \); hence \( \chi(O_{W'}) = 1 \) as desired.

Let us return to the original situation. By the argument in 1 we have \( \Sigma a_i < 2n + r \). Now one can apply the above claim to \( \mathbb{P}(X) \subset \mathbb{P}^{2n+r-1} \).

Q.E.D.

Since \( \mathbb{P}(X) \) has only rational singularities, we have \( \chi(Z, O_Z) = \chi(\mathbb{P}(X), O_{\mathbb{P}(X)}) = 1 \). Let us consider the projection map \( p : Z \to M \) of the projective space bundle. Since \( R^i p_* O_Z = 0 \) for \( i > 0 \), we have \( \chi(Z, O_Z) = \chi(M, O_M) \). In particular, we see that \( \chi(M, O_M) = 1 \).

Here we recall a special case of the theorem of Demailly, Peternell and Schneider [DPS]

**Theorem** ([DPS, Proposition on p.297]) : Let \( M \) be a projective manifold with nef tangent bundle such that \( \chi(M, O_M) \neq 0 \). Then \( M \) is a Fano manifold. When \( \dim M = 2 \) or 3, \( M \) is a rational homogeneous space.

In our case we have a much stronger condition. In fact, \( O_{\mathbb{P}(\Theta_M)}(1) \) is the pull-back of a very ample line bundle by a birational morphism.

**Proposition.** Let \( M \) be a Fano manifold. Assume that \( |O_{\mathbb{P}(\Theta_M)}(1)| \) is free from base points. Then \( M \) is isomorphic to a rational homogeneous space,
i.e. $M \cong G/P$ with a semisimple complex Lie group $G$ and its parabolic subgroup $P$.

Proof. The map $H^0(\mathbb{P}(\Theta_M), O_{\mathbb{P}(\Theta_M)}(1)) \otimes O_{\mathbb{P}(\Theta_M)}(1) \to O_{\mathbb{P}(\Theta_M)}(1)$ is surjective. Let us consider the natural map

$$H^0(M, \Theta_M) \otimes \Theta_M \to \Theta_M.$$  
We pull back $\alpha$ by the projection map $p : \mathbb{P}(\Theta_M) \to M$. Since $p_* O_{\mathbb{P}(\Theta_M)}(1) = \Theta_M$, $p^* \alpha$ factorizes $\beta$:

$$\beta : H^0(\mathbb{P}(\Theta_M), O_{\mathbb{P}(\Theta_M)}(1)) \otimes O_{\mathbb{P}(\Theta_M)}(1) \to p^* \Theta_M \to O_{\mathbb{P}(\Theta_M)}(1).$$

Let $x \in M$ be an arbitrary point and restrict $\beta$ to the fibre $p^{-1}(x) \cong \mathbb{P}^{n-1}$. Then we have

$$\beta(x) : H^0(\mathbb{P}(\Theta_M), O_{\mathbb{P}(\Theta_M)}(1)) \otimes O_{\mathbb{P}^{n-1}}(1) \to \mathbb{P}^{n-1} \to O_{\mathbb{P}^{n-1}}(1).$$

Note that $\beta(x)$ is also surjective. By taking the global sections $\beta(x)$ induces a map $\Gamma(\beta(x)) : H^0(\mathbb{P}(\Theta_M), O_{\mathbb{P}(\Theta_M)}(1)) \to H^0(\mathbb{P}^{n-1}, O_{\mathbb{P}^{n-1}}(1))$. If $\Gamma(\beta(x))$ is not surjective, then $\beta(x)$ cannot be surjective. Hence $\Gamma(\beta(x))$ must be surjective. This also shows that

$$\Gamma(p^* \alpha(x)) : H^0(\mathbb{P}(\Theta_M), O_{\mathbb{P}(\Theta_M)}(1)) \to H^0(\mathbb{P}^{n-1}, O_{\mathbb{P}^{n-1}}(1))$$

is surjective. Since $\Gamma(p^* \alpha(x))$ can be identified with the map $H^0(M, \Theta_M) \otimes k(x) \to \Theta_M \otimes k(x)$, the map $\alpha$ is a surjection by Nakayama’s lemma.

Let $G$ be the neutral component of the automorphism group $\text{Aut}(M)$ of $M$. Then $G$ can be written as the extension of a complex torus $T$ by a linear algebraic group $L$ (cf. [Fu])

$$1 \to L \to G \to T \to 1.$$  

Note that $q(M) = 0$ because $M$ is a Fano manifold. If $\dim T > 0$, then $\dim \text{Alb}(M) > 0$ by Theorem 5.5 of [Fu], which is a contradiction. Hence $G$ is a linear algebraic group. As $\alpha$ is surjective, $G$ acts transitively on $M$. Therefore $M \cong G/P$ for some parabolic subgroup $P$ of $G$ (cf. [Spr, 6.2]). Note that $P$ always contains the radical $r(G)$ of $G$. Then $r(G)$ acts trivially on $M$; but, since $G$ is the neutral component of $\text{Aut}(M)$, $G$ acts effectively on $M$. Hence $r(G) = \{1\}$ and $G$ is semisimple. Q.E.D.
5. Assume that \( n \geq 2 \). Now \( M \) can be written as \( G/P \) with \( G \) a semisimple complex Lie group and \( P \) a parabolic subgroup of \( G \). By the proof of the previous proposition we may assume that \( G = \text{Aut}^0(M) \). The cotangent bundle \( T^*(G/P) \) of \( G/P \) has a natural Hamiltonian \( G \)-action and one can define the moment map \( \mu : T^*(G/P) \to \mathfrak{g}^* \). We identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) by the Killing form. Then \( \text{Im}(\mu) \) coincides with the closure \( \bar{O} \) of a nilpotent orbit \( O \subset \mathfrak{g} \). The moment map induces a generically finite projective morphism of the projectivisations of \( T^*(G/P) \) and \( \bar{O} \):

\[
\bar{\mu} : P(\Theta_{G/P}) \to P(\bar{O}).
\]

Denote by \( O_{P(\bar{O})}(1) \) the restriction of the tautological line bundle \( O_{P(\mathfrak{g})}(1) \) of the projective space \( P(\mathfrak{g}) \) to \( P(\bar{O}) \). Then it can be checked that \( O_{P(\Theta_{G/P})}(1) = \bar{\mu}^*O_{P(\bar{O})}(1) \)

This means that \( \bar{\pi} : P(\Theta_{G/P}) \to P(X) \) must be the Stein factorization of \( \bar{\mu} \).

By looking at \( \bar{\mu} \) we have an inequality

\[
(1) \quad \dim \Gamma(P(\Theta_{G/P}), O_{P(\Theta_{G/P})}(1)) \geq \dim \Gamma(P(\bar{O}), O_{P(\bar{O})}(1)).
\]

Let \( I \) be the ideal sheaf of \( P(\bar{O}) \subset P(\mathfrak{g}) \). There is an exact sequence

\[
0 \to H^0(P(\mathfrak{g}), O_{P(\mathfrak{g})}(1) \otimes I) \to H^0(P(\mathfrak{g}), O_{P(\mathfrak{g})}(1)) \to H^0(P(\bar{O}), O_{P(\bar{O})}(1)).
\]

Let \( T_0\bar{O} \) be the tangent space of \( \bar{O} \) at the origin \( 0 \in \bar{O} \). Let \( \mathfrak{g} = \oplus \mathfrak{g}_i \) be the decomposition into the simple factors. The closure \( \bar{O} \) is the product of nilpotent orbit closures \( \bar{O}_i \) of \( \mathfrak{g}_i \). Note that \( T_0\bar{O} = \oplus T_0\bar{O}_i \). Each \( T_0\bar{O}_i \) is a sub \( G_i \)-representation of the adjoint \( G_i \)-representation of \( \mathfrak{g}_i \). Since \( \mathfrak{g}_i \) is an irreducible \( G_i \)-representation, we have \( T_0\bar{O}_i = \mathfrak{g}_i \). Hence \( T_0\bar{O} = \mathfrak{g} \).

This means that there is no hyperplane of \( \mathfrak{g} \) containing \( \bar{O} \); hence there is no hyperplane of \( P(\mathfrak{g}) \) containing \( P(\bar{O}) \). This shows that \( H^0(P(\mathfrak{g}), O_{P(\mathfrak{g})}(1) \otimes I) = 0 \). Since \( h^0(P(\mathfrak{g}), O_{P(\mathfrak{g})}(1)) = \dim \mathfrak{g} \), we have an inequality

\[
(2) \quad \dim \Gamma(P(\bar{O}), O_{P(\bar{O})}(1)) \geq \dim \mathfrak{g}.
\]

\[3\]Let \( \omega_{KK} \) be the Kostant-Kirillov 2-form on \( O \). Then it gives a contact structure on \( P(O) \) with the contact line bundle \( O_{P(O)}(1) \). On the other hand, \( \mu^*\omega_{KK} \) is a symplectic form on \( T^*(G/P) \), which gives a contact structure on \( P(\Theta_{G/P}) \) with the contact line bundle \( \bar{\mu}^*O_{P(\bar{O})}(1) \). Then we can apply [KPSW, Theorem 2.12] to conclude that \( \bar{\mu}^*O_{P(\bar{O})}(1) = O_{P(\Theta_{G/P})}(1) \).
By (1) and (2) we have an inequality
\[ \dim \Gamma(P(\Theta_{G/P}), O_{P(\Theta_{G/P})}(1)) \geq \dim \mathfrak{g}. \]

Since \( \Gamma(P(\Theta_{G/P}), O_{P(\Theta_{G/P})}(1)) = \Gamma(G/P, \Theta_{G/P}(1)) \), this inequality is actually an equality. Hence \( \bar{\pi} \) coincides with \( \bar{\mu} \) and we have an isomorphism of polarised varieties \( (P(X), O_{P(X)}(1)) \sim (P(\bar{O}), O_{P(\bar{O})}(1)) \). As \( X = \text{Spec} \oplus_{m \geq 0} H^0(P(X), O_{P(X)}(m)) \) and \( \bar{O} = \text{Spec} \oplus_{m \geq 0} H^0(P(O), O_{P(O)}(m)) \), this implies that \( X = \bar{O} \).

Finally we give an intrinsic characterization of \( G \). Notice that we have taken an isomorphism \( Z \cong P(\Theta_M) \) such that the contact structure corresponds to the canonical one induced by the canonical 2-form on \( T^*M \). Then \( G \) acts on \( Z \) as contact automorphisms. Since \( \bar{\pi} \) is \( G \)-equivariant, this also means that \( G \) acts on \( P(X)_{\text{reg}} \) as contact automorphisms. The \( G \)-action determines an embedding \( \mathfrak{g} \subset H^0(P(X)_{\text{reg}}, \Theta_{P(X)_{\text{reg}}}) \).

On the other hand, by [LeB] the contact structure
\[ \Theta_{P(X)_{\text{reg}}} \rightarrow O_{P(X)}(1)|_{P(X)_{\text{reg}}} \rightarrow 0 \]
has a splitting (as \( \mathbb{C} \)-modules)
\[ s : O_{P(X)}(1)|_{P(X)_{\text{reg}}} \rightarrow \Theta_{P(X)_{\text{reg}}} \]
so that the subspace
\[ s(H^0(P(X)_{\text{reg}}, O_{P(X)}(1)|_{P(X)_{\text{reg}}}) \subset H^0(P(X)_{\text{reg}}, \Theta_{P(X)_{\text{reg}}}) \]
is the infinitesimal contact automorphism group of \( P(X)_{\text{reg}} \). By the observation above it has the same dimension as \( \dim \mathfrak{g} \). Hence \( \mathfrak{g} \subset H^0(P(X)_{\text{reg}}, \Theta_{P(X)_{\text{reg}}}) \) coincides with the infinitesimal contact automorphism group of \( P(X)_{\text{reg}} \) (or \( P(X) \)) and \( G \) is the neutral component of the contact automorphism group of \( P(X) \).

We have thus proved:

**Theorem 2.** Let \( X \) be a singular symplectic variety embedded in an affine space \( \mathbb{C}^N \) as a complete intersection of homogeneous polynomials. Then \( X \) coincides with a nilpotent orbit closure \( \bar{O} \) of a semisimple complex Lie algebra \( \mathfrak{g} \).

By the proof such an orbit \( O \) is a Richardson orbit and the Springer map \( T^*(G/P) \rightarrow \bar{O} \) is a birational map.
A typical example of $\mathcal{O}$ is the nilpotent variety $N$ of $\mathfrak{g}$. Let $\chi : \mathfrak{g} \to \mathfrak{g}/G = \mathbb{C}^r$ be the adjoint quotient map. Then $N = \chi^{-1}(0)$. In particular, $N$ is a complete intersection of $r$ homogeneous polynomials in $\mathfrak{g}$.

The following is the main theorem of this article.

**Main Theorem.** Let $(X, \omega)$ be a singular symplectic variety embedded in an affine space $\mathbb{C}^N$ as a complete intersection of homogeneous polynomials. Assume that $\omega$ is also homogeneous. Then $(X, \omega)$ coincides with the nilpotent variety $(\mathcal{N}, \omega_{KK})$ of a semisimple complex Lie algebra $\mathfrak{g}$ together with the Kostant-Kirillov form $\omega_{KK}$.

6. In this section we prove that the nilpotent orbit closure $\mathcal{O}$ in Theorem 2 is actually the nilpotent variety $N$.

(6.1) Let $\mathbb{C}[x_1, ..., x_n]$ be a polynomial ring with $n$ variables. For a homogeneous ideal $I$ of $\mathbb{C}[x_1, ..., x_n]$, we put $R := \mathbb{C}[x_1, ..., x_n]/I$ and $d := \dim R$. Assume that $I$ does not contain a non-zero homogeneous polynomial of degree 1. We denote by $M$ the maximal ideal $(x_1, ..., x_n)$ of $R$.

**Lemma.** The following are equivalent.

(i) The formal completion $\hat{R}$ along $M$ is of complete intersection.

(ii) The ideal $I$ is generated by $n - d$ homogeneous elements.

**Proof.** Since it is clear that (ii) implies (i), we only have to prove that (i) implies (ii). The number of minimal generators of $\hat{I}$ equals $\dim \mathbb{C}(I/IM)$ by Nakayama’s lemma. The condition (i) then means that $\dim \mathbb{C}(I/IM) = n - d$. One can take $n - d$ homogeneous elements $f_1, ..., f_{n-d}$ from $I$ such that $\bar{f}_1, ..., \bar{f}_{n-d}$ form a basis of $I/IM$. Then it can be checked that $f_1, ..., f_{n-d}$ actually generate $I$ (cf. the proof of Lemma (A.4) of [Na 1]). Q.E.D.

(6.2) Let $R$ be the same as in (6.1) and put $X := \text{Spec}(R)$. Assume that a reductive Lie group $G$ acts on $\mathbb{C}^n = \text{Spec}\mathbb{C}[x_1, ..., x_n]$ so that $X$ is preserved by $G$. Moreover we assume that the $G$-action commutes with the $\mathbb{C}^*$-action on $\mathbb{C}^n$.

**Lemma.** There are a $G$-representation $V$ with $\dim V = n - d$ and a $G$-equivariant morphism $f : \mathbb{C}^n \to V$ of affine spaces such that $f^{-1}(0) = X$.

**Proof.** Let $I_k$ be the degree $k$ part of the homogeneous ideal $I$. Since $G$ respects the grading of $\mathbb{C}[x_1, ..., x_n]$, each $I_k$ is a $G$-representation. Let $k_1$ be the minimal number such that $I_{k_1} \neq 0$. Let $k_2$ be the minimal number $k > k_1$ such that $I'_k := C[x_1, ..., x_n]_{k-k_1} \cdot I_{k_1}$ does not coincide with $I_k$. Since $I'_{k_2}$ is a $G$-subrepresentation of $I_{k_2}$, there is a $G$-subrepresentation $I''_{k_2}$ of $I_{k_2}$ such that $I_{k_2} = I'_{k_2} \oplus I''_{k_2}$. We next put $I_k' := C[x_1, ..., x_n]_{k-k_2} \cdot I_{k_2}$ for $k > k_2$.
and let $k_3$ be the minimal number $k$ such that $I'_k \neq I_k$. Let $I''_{k_3}$ be a $G$-subrepresentation of $I_{k_3}$ such that $I_{k_3} = I'_{k_3} \oplus I''_{k_3}$. We repeat this process; then $I_{k_1} \oplus I''_{k_2} \oplus I''_{k_3} \oplus \ldots$ becomes a $G$-representation of dimension $n - d$. The $V$ is its dual representation. Q.E.D.

(6.3) **Proposition.** A nilpotent orbit closure $\bar{O}$ of an exceptional simple Lie algebra $\mathfrak{g}$ is of complete intersection if and only if $\bar{O} = N$.

**Proof.** We put $m := \dim \mathfrak{g}$ and $2n := \dim \bar{O}$. Then $\bar{O}$ is an affine subvariety of $\mathbb{C}^m$ with codimension $r := m - 2n$. Assume that $\bar{O}$ is defined by $r$ homogeneous polynomials $f_i$ with $\deg(f_i) = a_i$. As remarked at the beginning of 1, we have $\Sigma_{1 \leq i \leq r} a_i = n + r$. Since $a_i \geq 2$ for all $i$, we see that $\Sigma a_i \geq 2r$; thus $n \geq r$. In particular, $m = 2n + r \geq 3r$. Therefore we have

$$\text{Codim}_g \bar{O} \leq 1/3 \cdot \dim \mathfrak{g}.$$ 

On the other hand, by the previous lemma there are a $G$-representation $V$ with $\dim V = \text{Codim}_g \bar{O}$ and a $G$-equivariant map $f : \mathfrak{g} \to V$ such that $f^{-1}(0) = \bar{O}$. There are very few (nontrivial) irreducible representations $V$ of an exceptional simple Lie group $G$ with $\dim V < \dim G$ (cf. [F-H], Exercise 24.52 (p.414, see also pp.531,532). These are:

- $G_2$: $\dim \mathfrak{g} = 14$, $\dim V_{\omega_1} = 7$,
- $F_4$: $\dim \mathfrak{g} = 52$, $\dim V_{\omega_4} = 26$,
- $E_6$: $\dim \mathfrak{g} = 78$, $\dim V_{\omega_1} = \dim V_{\omega_6} = 27$,
- $E_7$: $\dim \mathfrak{g} = 133$, $\dim V_{\omega_7} = 56$.

Here we denote by $V_{\omega_i}$ the representations $\Gamma_{\omega_i}$ in [F-H]. As a consequence, we have no irreducible representation $V$ with $\dim V \leq 1/3 \cdot \dim \mathfrak{g}$. Let us look at the $G$-equivariant map $f : \mathfrak{g} \to V$. Since there is no irreducible $G$-representation of $\dim \leq 1/3 \cdot \dim \mathfrak{g}$, the $G$-representation $V$ is a direct sum of trivial representations. This means that $\bar{O}$ is the common zeros of some invariant polynomials on $\mathfrak{g}$ (with respect to the adjoint representation). Notice that the nilpotent variety $N$ of $\mathfrak{g}$ is the common zeros of all invariant polynomials on $\mathfrak{g}$. Since $\bar{O}$ is contained in $N$, we conclude that $\bar{O} = N$. Q.E.D.

(6.4) Let $G$ be a semisimple complex Lie group and let $P$ be a parabolic subgroup of $G$. Let $O \subset \mathfrak{p}$ be the Richardson orbit for $P$. We assume that the closure $\bar{O}$ is normal and the Springer map $T^*(G/P) \to \bar{O}$ is birational. One can construct a flat deformation of $\bar{O}$ in the following way. Details can be found in [Na 4, Section 2]. Let $n(\mathfrak{p})$ (resp. $r(\mathfrak{p})$) be the nilradical (resp. solvable radical) of $\mathfrak{p}$. Let $\mathfrak{h} \subset \mathfrak{p}$ be a Cartan subalgebra of $\mathfrak{p}$ and define
\[ \mathfrak{k}(p) := \mathfrak{h} \cap r(p). \] We then have \( r(p) = \mathfrak{k}(p) \oplus n(p). \) Notice that \( \bar{O} \) is the \( G \)-orbit of \( n(p) \):

\[ \bar{O} = G \cdot n(p). \]

Then \( G \cdot r(p) \) naturally contains \( \bar{O} \). Restricting the adjoint quotient map \( \chi : g \to \mathfrak{h}/W \) to \( G \cdot r(p) \), we have a map

\[ \chi_p : G \cdot r(p) \to \mathfrak{h}/W. \]

Let \( \nu : X \to G \cdot r(p) \) be the normalization map. Then the composition map \( X \to \mathfrak{h}/W \) factors through \( k(p)/W' \), where \( W' \subset W \) is the stabilizer subgroup of \( k(p) \):

\[ \chi_n^p : X \to k(p)/W'. \]

By [Na 4, Proposition 2.6] we have \( (\chi_n^p)^{-1}(0) = \bar{O} \) and \( \chi_n^p \) gives a flat deformation of \( \bar{O} \). There is a natural \( \mathbb{C}^* \)-action on \( X \). If \( (\chi_n^p)^{-1}(0) = \bar{O} \) is of locally complete intersection, then all fibres \( (\chi_n^p)^{-1}(t) \) are also of locally complete intersection by the \( \mathbb{C}^* \)-action.

(6.5) A fibre of \( \chi_p \) has been already studied in [Sl, 4.3]. For \( t \in \mathfrak{h} \) define \( Z_G(t) \subset G \) to be the centralizer of \( t \) in \( G \); namely

\[ Z_G(t) := \{ g \in G; Ad_g(t) = t \}. \]

Similarly define \( Z_g(t) \subset \mathfrak{g} \) to be the centralizer of \( t \) in \( \mathfrak{g} \). Note that \( Z_g(t) \) is a reductive Lie algebra. Then \( p_t := \mathfrak{p} \cap Z_G(t) \) is a parabolic subalgebra of \( Z_g(t) \). Let \( O_t \subset Z_g(t) \) be the Richardson orbit for \( p_t \). Take an element \( \tilde{t} \) from the image of the map \( k(p) \to \mathfrak{h}/W \). Then the fibre \( \chi_n^p(t) \) can be described as follows. Let \( \{t_1, ..., t_n\} \) be the inverse image of \( \tilde{t} \) by the map \( k(p) \to \mathfrak{h}/W \). Then one has

\[ \chi_n^p(t) = \bigcup_{1 \leq i \leq n} \rho_i(G \times Z_G(t_i) (t_i + \tilde{O}_{t_i})), \]

where \( \rho_i : G \times Z_G(t_i) (t_i + \tilde{O}_{t_i}) \to G \cdot r(p) \) is a map defined by \( \rho_i([g, t_i + x]) = Ad_g(t_i + x) \). As remarked in [Sl, p.56, Remark], \( \chi_n^p(t) \) is not necessarily irreducible. However a fibre of \( \chi_n^p \) is always irreducible and normal. Consider the Brieskorn-Slodowy diagram ([Na 4, p.728 (2)]):

\[ \begin{array}{ccc}
G \times^B r(p) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathfrak{e}(p) & \longrightarrow & \mathfrak{e}(p)/W'
\end{array} \]
Here \( G \times^P r(p) \) gives a simultaneous resolution of the flat family \( \mathcal{X} \times_{t(p)/W'} \mathfrak{k}(p) \to \mathfrak{g}(p) \). Take an element \( t \) from \( \mathfrak{k}(p) \). The fibre of the map \( G \times^P r(p) \to \mathfrak{k}(p) \) over \( t \) is \( G \times^P (t + n(p)) \). Notice that

\[
G \times^P (t + n(p)) = G \times^P (P \times^P (t + n(p))) = G \times^P (t + n(p)) = G \times^P (t + n(p)),
\]

Let \( \bar{t} \in \mathfrak{k}(p)/W' \) be the image of \( t \) by the map \( \mathfrak{k}(p) \to \mathfrak{k}(p)/W' \). Then the map

\[
G \times^P (t + n(p)) \to \mathcal{X}_{\bar{t}}
\]

coincides with the map

\[
G \times^P (Z_G(t) \times^P (t + n(p))) \to G \times^P (t + \bar{O}_t),
\]

where \( \bar{O}_t \) is the normalization of the orbit closure \( \bar{O}_t \). In particular, one has

\[
(\chi^n_p)^{-1}(\bar{t}) = G \times^P (t + \bar{O}_t) \cong G \times^P (t + \bar{O}_t).
\]

Note that \( (\chi^n_p)^{-1}(\bar{t}) \) is locally the product of \( G/Z_G(t) \) and \( \bar{O}_t \). If the central fibre \( (\chi^n_p)^{-1}(0) \) is locally of complete intersection, then \( \bar{O}_t \) is locally of complete intersection.

(6.6) Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). Let \( \Phi \) be the root system for \( \mathfrak{g} \). Choose a base \( \Delta \) of \( \Phi \). Recall that every parabolic subgroup of \( G \) is conjugate to a standard parabolic subgroup \( P_I \) for a subset \( I \) of \( \Delta \). We denote by \( L(P_I) \) the Levi subgroup of \( P_I \) containing \( H \). For example, if \( I = \emptyset \), then \( P_I \) is a Borel subgroup and \( L(P_I) \) is nothing but the maximal torus \( H \) of \( G \). In the remainder we assume that \( P \) is a standard one \( P_I \). One has

\[
\mathfrak{k}(p_I) = \{ h \in \mathfrak{h}; \alpha(h) = 0, \forall \alpha \in I \}.
\]

Define

\[
\mathfrak{k}(p_I)_{\text{reg}} := \{ h \in \mathfrak{k}(p_I); \alpha(h) \neq 0, \forall \alpha \in \Phi_I \}
\]

where \( \Phi_I \) is the root subsystem of \( \Phi \) generated by \( I \). Choose \( \beta \in \Delta - I \) and consider the larger parabolic subgroup \( P_{I \cup \{\beta}\}} \). Then \( \mathfrak{k}(p_{I \cup \{\beta}\})_{\text{reg}} \) is naturally contained in \( \mathfrak{k}(p_I) \). We take an element \( t_\beta \) from \( \mathfrak{k}(p_{I \cup \{\beta}\})_{\text{reg}} \). Notice that \( Z_G(t_\beta) = L(P_{I \cup \{\beta}\}) \). Moreover \( P_I \cap Z_G(t_\beta) \) is a parabolic subgroup of \( Z_G(t_\beta) \), which determines a Richardson orbit \( O_{t_\beta} \) of \( Z_G(t_\beta) \). We then have

\[
(\chi^n_{p_I})^{-1}(\bar{t}_\beta) \cong G \times^P (t_\beta) \bar{O}_{t_\beta}.
\]
(6.7) Example. Let $P_I$ be the standard parabolic subgroup of $SL(5)$ determined by the following marked Dynkin diagram, where the white vertices are simple roots belonging to $I$:

![Dynkin diagram]

We have two black vertices. Take the 1-st black vertex as $\beta$. Then the semisimple reduction $[Z_g(t_\beta), Z_g(t_\beta)]$ is of type $A_2 \times A_1$. Moreover $O_{t_\beta}$ is the Richardson orbit of the first $A_2$ for the parabolic subalgebra corresponding to

![Next diagram]

Next take the 2-nd black vertex as $\beta$. Then $[Z_g(t_\beta), Z_g(t_\beta)]$ is of type $A_3$. The orbit $O_{t_\beta}$ is the Richardson of $A_3$ for the parabolic subalgebra corresponding to

![Another diagram]

Let $O \subset sl(5)$ be the Richardson orbit for $P_I$. Assume that $\bar{O}$ is locally of complete intersection. Then $\bar{O}_t$ is locally of complete intersection for any $t \in \mathfrak{p}_I$ by (6.5). As above we take the 1-st black vertex as $\beta$ and consider the corresponding $O_{t_\beta}$. It is then easily checked that $\text{Codim}_{\bar{O}_t} \text{Sing}(\bar{O}_t) = 4$. By [Be, Proposition 1.4] $\bar{O}_t$ is not locally of complete intersection. This is absurd. The second choice of $\beta$ also leads us to a contradiction. In this case $\text{Sing}(\bar{O}_{t_\beta})$ has codimension 2 in $\bar{O}_{t_\beta}$ and Beauville’s proposition cannot be used. Instead we use the previous lemma. First notice that every nilpotent orbit closure in $sl(m)$ is normal; hence $\bar{O}_{t_\beta} = \bar{O}_{t_\beta}$. By a direct calculation one has $\dim \bar{O}_{t_\beta} = 8$ and $\dim sl(4) = 15$. Suppose that $\bar{O}_{t_\beta}$ is locally of complete intersection. As proved in 5, $T_0 \bar{O}_{t_\beta} = sl(4)$; one can apply Lemma (6.1) to the embedding $\bar{O}_{t_\beta} \subset sl(4)$. Then $\bar{O}_{t_\beta}$ is defined as the common zeros of 7 homogeneous polynomials $f_i$ ($1 \leq i \leq 7$). We put $a_i := \deg(f_i)$. By the argument at the beginning of 1 we have $a_1 + \ldots + a_7 = 11$. On the other hand, since $a_i \geq 2$ for all $i$, we have $a_1 + \ldots + a_7 \geq 14$. This is a contradiction.

(6.8) We are now going to prove that when $\mathfrak{g}$ is a classical simple Lie algebra, the nilpotent orbit closure $\bar{O}$ in Theorem 2 is actually the nilpotent variety $N$. We employ the following strategy. We shall derive a contradiction assuming that $\bar{O}$ in Theorem 2 is not the nilpotent variety. First we construct a flat deformation of $\bar{O}$: $\chi^\alpha : \mathcal{X} \rightarrow \mathfrak{k}/W'$ as in (6.4). The parabolic sub-
algebra $\mathfrak{p}$ corresponds to a marked Dynkin diagram for $\mathfrak{g}$. As demonstrated in (6.7), we take a suitable simple root $\beta$ and the corresponding element $t_\beta \in \mathfrak{g}(\mathfrak{p})$ (cf. (6.6)). We next consider the fibre $(\chi^\mathfrak{p}_\beta)^{-1}(t_\beta)$. Then this fibre is isomorphic to $G \times_{\mathbb{Z}G(t_\beta)} \tilde{O}_{t_\beta}$. If $\tilde{O}$ is of complete intersection, then $\tilde{O}_{t_\beta}$ is also of complete intersection. But $O_{t_\beta}$ is a Richardson orbit in a classical simple Lie algebra which is smaller than $\mathfrak{g}$. Moreover the corresponding parabolic subalgebra ($= \text{the polarization of } O_{t_\beta}$) is a maximal parabolic subalgebra. Finally we derive a contradiction in such a case.

We first treat the case $\mathfrak{g}$ is of type $A$.

**Proposition.** A nilpotent orbit closure $\bar{O}$ of $\mathfrak{sl}(m)$ has complete intersection singularities if and only if $\bar{O} = N$.

**Proof.** Note that every nilpotent orbit $O$ of $\mathfrak{g} := \mathfrak{sl}(m)$ is a Richardson orbit and its closure is normal. Moreover the Springer map $T^*(G/P) \to \bar{O}$ is birational. As remarked just above, we only have to prove that $\bar{O}$ does not have complete intersection singularities when $P$ is a maximal parabolic subgroup of $SL(m)$ with $m \geq 3$. Namely $P$ corresponds to to a marked Dynkin diagram with only one black vertex:

$$
\begin{array}{c}
1 \quad - - - \quad r \quad - - - \quad \circ
\end{array}
$$

When $r \neq m/2$, one has $\text{Codim}_O \text{Sing}(\bar{O}) \geq 4$. Then $\bar{O}$ does not have complete intersection singularities by [Be, Proposition 1.4]. Assume that $\bar{O}$ has complete intersection singularities when $r = m/2$. By a direct calculation we have $\dim \bar{O} = 2r^2$ and $\dim \mathfrak{sl}(m) = 4r^2 - 1$. By Lemma (6.1) $\bar{O}$ is a subvariety of $\mathbb{C}^{4r^2 - 1}$ defined as the complete intersection of $2r^2 - 1$ homogeneous polynomials $f_i$. We put $a_i := \deg(f_i)$. As discussed at the beginning of $1$, $\Sigma a_i = r^2 + (2r^2 - 1)$. On the other hand, since $a_i \geq 2$, we have $\Sigma a_i \geq 2(2r^2 - 1)$. Combining these inequalities we get

$$
r^2 \geq 2r^2 - 1,
$$

which implies that $r = 1$ and then $m = 2$. This contradicts the first assumption that $m \geq 3$. Q.E.D.

(6.9) Let $G$ be $Sp(2n)$ or $SO(n)$ and let $P_I$ be a maximal parabolic subgroup. Namely $P$ is the standard parabolic subgroup corresponding to one of the following Dynkin diagram.

$$
\begin{array}{c}
1 \quad - - - \quad r \quad - - - \quad \circ
\end{array}
$$
Let $O \subset \mathfrak{g}$ be the Richardson orbit for $P_I$. We shall prove that $\bar{O}$ is not a homogeneous symplectic variety of complete intersection. When $G = Sp(2n)$, the parabolic subgroup $P_I$ is the stabilizer group of an isotropic flag of type $(r, 2n - r, r)$. Let $Gr_{iso}(r, 2n)$ be the isotropic Grassmann variety parametrizing such flags. Then

$$\dim Gr_{iso}(r, 2n) = \dim Gr(r, 2n) - 1/2 \cdot r(r-1) = r(2n - r) - 1/2 \cdot r(r-1).$$

Since $\dim \bar{O} = 2 \dim Gr_{iso}(r, 2n)$, we have $\dim \bar{O} = 2r(2n - r) - r(r-1)$. On the other hand, $\dim sp(2n) = 2n^2 + n$, hence $\text{Codim}_{sp(2n)} \bar{O} = 2n^2 + n - 4rn + 3r^2 - r$. Assume $\bar{O}$ is of complete intersection in $sp(2n)$. Let $f_i$ be the defining equations of $\bar{O}$ and put $a_i := \deg(f_i)$. Then $\sum a_i = 1/2 \cdot \dim \bar{O} + \text{Codim}_{sp(2n)} \bar{O}$ by 1. Since $a_i \geq 2$ for all $i$, we have $(3r - 2n - 1)(3r - 2n) \leq 0.$

The only possibilities are following two cases:

(i) $n = 3k$ for some integer $k$ and $r = 2k$.

(ii) $n = 3k + 1$ for some integer $k$ and $r = 2k + 1$.

In both cases $a_i = 2$ for all $i$ (i.e. $\dim V = 1/3 \cdot \dim sp(2n)$.) In the first case $O = O_{[3\kappa]}$ (i.e the nilpotent orbit consisting of the matrices of Jordan type $(3, ..., 3)$ $(2k$ Jordan blocks of size 3$). In the second case $O = O_{[3\kappa, 2]}$. Assume that $O_{[3\kappa]} \subset sp(6k)$ is of complete intersection. By the calculation above we have $\text{codim}_{sp(6k)} \bar{O} = 6k^2 + k$. By Lemma (6.2) there are a $G$-representation $V$ of dim $6k^2 + k$ and a $G$-equivariant map $f : sp(6k) \rightarrow V$ such that $f^{-1}(0) = \bar{O}$. By the construction of $V$ (cf. Lemma (6.2)), the dual representation $V^*$ coincides with $I_2$ because $a_i = 2$ for all $i$. But there is only one (adjoint) invariant quadratic polynomial on $sp(6k)$ up to constant. Hence $V$ contains one and only one trivial representation as a direct factor. Since an irreducible representation of $sp(6k)$ with dim $\leq 1/3 \cdot \dim sp(6k)$ is a trivial representation or a standard representation (cf. [F-H], p.531, (24.52)), $V$ is a direct sum of a trivial representation and a finite number of standard representations.

Let us consider the first case (i). Notice that, in this case, $\dim V = 1 + (6k^2 + k - 1)$. If $k \geq 2$, then $6k$ does not divide $6k^2 + k - 1$, which is
a contradiction. When $k = 1$, one has $\dim V = 7$ and $V$ may possibly be a direct sum of the 6-dimensional standard representation and the trivial representation. Since $a_i = 2$ for all $i$, these irreducible factors must be contained in $\text{Sym}^2(sp(6)^*)$ the 2-nd symmetric product of the dual representation of the adjoint one. By the Killing form $\text{Sym}^2(sp(6)^*) \cong \text{Sym}^2(sp(6))$ as $Sp(6)$-representations. It is easily checked that $\text{Sym}^2(sp(6))$ does not contain the standard representation as a direct factor. Hence we have a contradiction also in this case.

In the second case (ii) we have $\dim V = 1 + (6k^2 + 5k)$. Noticing that the standard representation has dimension $6k + 2$, we write $6k^2 + 5k = k(6k + 2) + 3k$; hence $6k + 2$ does not divide $6k^2 + 5k$. This is a contradiction.

Assume that $G = SO(n)$ and $\bar{O}$ has complete intersection singularities. Since $a_i \geq 2$ for all $i$, the equality

$$\sum a_i = 1/2 \cdot \dim \bar{O} + \text{Codim}_{so(n)}\bar{O}$$

implies that $(3r - n)(3r - n + 1) \leq 0$. There are two possibilities:

(i) $n = 3k$ for some integer $k$, $r = k$ and $O = O_{[3k]}$.

(ii) $n = 3k + 1$ for some integer $k$, $r = k$ and $O = O_{[3k+1]}$.

In both cases $a_i = 2$ for all $i$ (i.e. $\dim V = 1/3 \cdot \dim so(n)$). We can again use Lemma (6.2) to have a $G$-equivariant map $f : so(n) \to V$. Put $\mathfrak{g} = so(n)$ with $n = 3k$ or $n = 3k + 1$. Then $\dim V$ is respectively $1/2 \cdot (3k^2 - k)$ or $1/2 \cdot (3k^2 + k)$. Note that an irreducible representation of $\mathfrak{g}$ with dim $\leq 1/3 \cdot \dim \mathfrak{g}$ is a trivial representation or a standard representation (cf. [F-H], p.531, (24.52): Note that, when $\mathfrak{g}$ is of $D_4$, two more different irreducible representations exist, but the $D_4$ case is not contained in the case (i) or the case (ii).). Since there is only one (adjoint) invariant quadratic polynomial on $so(n)$ up to constant, $V$ is a direct sum of a trivial representation and a finite number of standard representations. By writing $k = 2l$ or $k = 2l + 1$ according as $k$ is even or odd, one can easily check that $\dim V - 1$ is not divided by $n$ in both cases; hence we have a contradiction.

(6.10) Let $\mathfrak{g}$ be a complex simple Lie algebra of type $B$, $C$ or $D$. Let $O$ be the Richardson orbit of $\mathfrak{g}$ for a parabolic subgroup $P$ of $G$. Assume that the Springer map $s : T^*(G/P) \to \bar{O}$ is birational.

**Proposition** The closure $\bar{O}$ of such an orbit is of complete intersection if and only if $\bar{O} = N$.

**Proof.** We only have to deal with a Richardson orbit for a standard parabolic subgroup $P_I$. If the Dynkin diagram corresponding to $P_I$ has only
one black vertex, then we have already checked that $\mathcal{O}$ is not of complete intersection. Assume that there are more than one black vertices, but at least one vertex is a white vertex. Take a white vertex $w$ on the leftmost position. Note that if the Dynkin diagram is of type $B$ or $C$, it is unique, but if the Dynkin diagram is of type $D$, the choice of such a vertex might have two possibilities.

If there is a black vertex $b$ left adjacent to $w$, then take the simple root $\beta$ corresponding to $b$ and apply (6.6). Then the problem is reduced to the case where the Dynkin diagram is of type $A$ and has only one black vertex with $r = 1$, or the Dynkin diagram is a smaller one of the same type as $\mathfrak{g}$ and has only one black vertex with $r = 1$. In each case $\mathcal{O}_{t_o}$ is normal; we only have to check this in the second case. There is a nilpotent orbit $\mathcal{O}'_{t_o} \subset \mathcal{O}_{t_o}$ such that $\text{Codim} \mathcal{O}_{t_o} \mathcal{O}'_{t_o} = 2$. One can check that $\text{Sing}( \mathcal{O}_{t_o}, \mathcal{O}'_{t_o})$ is of type $a$ or of type $g$ in the list of [K-P, p.551]. By Theorem 1, (b) of [K-P] we see that $\mathcal{O}_{t_o}$ is normal. Moreover, in each case, $\mathcal{O}_{t_o}$ is not of complete intersection (cf. (6.8), (6.9)). By the argument in (6.5), the original nilpotent orbit closure $\mathcal{O}$ is not of complete intersection.

Assume that there is no black vertex left adjacent to $w$. By the definition of $w$ this means that $w$ is on the leftmost position on the diagram. In this case we consider the maximal connected Dynkin subdiagram $\mathcal{D}$ containing $w$ whose vertices are all white. Let $w'$ be a vertex on the rightest position of $\mathcal{D}$. Let $b$ be a black vertex right adjacent to $w'$. We take the simple root $\beta$ corresponding to $b$ and apply (6.6). Then the problem is reduced to the case where the Dynkin diagram is of type $A$ and has only one black vertex. Then $\mathcal{O}_{t_o}$ is normal and is not of complete intersection (cf. (6.8)). By the argument in (6.5), the original nilpotent orbit closure $\mathcal{O}$ is not of complete intersection. Q.E.D.

(6.11) Let $\mathcal{O}$ be a Richardson orbit of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{1 \leq i \leq m} \mathfrak{g}_i$ be the decomposition into the simple factors. Then we have $\mathcal{O} = \mathcal{O}_1 \times \ldots \times \mathcal{O}_m$ where each $\mathcal{O}_i$ is a Richadson orbit of $\mathfrak{g}_i$. If the Springer map $T^*(G/P) \to \mathcal{O}$ is birational, then each Springer map $T^*(G_i/P_i) \to \mathcal{O}_i$ is birational. Assume that $\mathcal{O}$ is of complete intersection. Then each $\mathcal{O}_i$ is also of complete intersection. By (6.3), (6.8) and (6.10) each $\mathcal{O}_i$ coincides with the nilpotent variety $\mathcal{N}_i$ of $\mathfrak{g}_i$. Then $\mathcal{O}$ is the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$.

7. Remarks

(1) What happens in Main theorem if we do not assume $\omega$ is homogeneous? The author does not know the answer, but the following example would
be instructive. Let $X \subset \mathbb{C}^5$ be a hypersurface defined by $z_1^2 + z_2^2 + z_3^2 = 0$, where $(z_1, ..., z_3)$ are coordinates of $\mathbb{C}^5$. Note that $X = S \times \mathbb{C}^2$, where $S \subset \mathbb{C}^5$ is a hypersurface defined by $f := z_1^2 + z_2^2 + z_3^2 = 0$. We put $\omega_S := \text{Res}(dz_1 \wedge dz_2 \wedge dz_3/f)$ and $\omega_{\mathbb{C}^2} := dz_4 \wedge dz_5$. Define $\omega := \omega_S + \omega_{\mathbb{C}^2}$. Then $(X, \omega)$ is an affine symplectic variety. But $\omega$ is not homogeneous because $wt(\omega_S) = 1$ and $wt(\omega_{\mathbb{C}^2}) = 2$. Note that $\omega \wedge \omega$ is a holomorphic volume form on $X$ of weight $3$. Let us prove that there is no homogeneous symplectic 2-form on $X$. Assume that such a form $\Omega$ exists. Then $\Omega \wedge \Omega$ is a holomorphic volume form on $X$ of an even weight, say $2m$. Then one can write $\Omega \wedge \Omega = g \cdot \omega \wedge \omega$ with a nowhere vanishing function $g$ of nonzero weight. But such $g$ does not exist; hence one gets a contradiction.

(2) Let $X$ be an affine symplectic variety in $\mathbb{C}^N$ defined by a homogeneous ideal $I$ (not necessarily of complete intersection) where $I$ contains no nonzero linear form. Denote by $R$ the coordinate ring of $X$. By the assumption $R$ is graded: $R = \oplus_{n \geq 0} R_n$. Assume that $wt(\omega) = 1$. Then $\omega$ induces a Poisson structure on $R$ of weight $-1$. In particular, it induces a Lie algebra structure on $R_1$

$$[\cdot, \cdot] : R_1 \times R_1 \rightarrow R_1.$$ 

Let us call this Lie algebra $\mathfrak{g}$. Since $R_1 = T_0^*X$, we have $\dim \mathfrak{g} = N$. The natural surjection $\oplus \text{Sym}^i(R_1) \rightarrow R$ induces a closed embedding $X \rightarrow \mathfrak{g}^*$. To prove that $\mathfrak{g}$ is semisimple, it seems that one needs some geometric arguments as in 1 - 5. When $\mathfrak{g}$ is semisimple, $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ by the Killing form. This is nothing but the closed embedding $X \rightarrow \mathfrak{g}$ of Main theorem, where $X$ is identified with the nilpotent variety $N$.

(3) Let $X$ be the same as in (2). Then $\mathbf{P}(X)$ admits a contact structure with the contact line bundle $O_{\mathbf{P}(X)}(1)$ in the sense of 1. Let $G$ be the contact automorphism group of $\mathbf{P}(X)_{\text{reg}}$. The Lie algebra $\mathfrak{g}$ is contained in $H^0(\mathbf{P}(X), \Theta_{\mathbf{P}(X)})$ and the map $H^0(\mathbf{P}(X), \Theta_{\mathbf{P}(X)}) \rightarrow H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(1))$ induces an isomorphism $\mathfrak{g} \cong H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(1))$ by [Be 2], Proposition 1.1. In general we only know that $\dim \mathfrak{g} \geq N$. The closed embedding $\mathbf{P}(X) \rightarrow \mathbf{P}(\mathfrak{g}^*)$ is a $G$-equivariant map. By a similar argument to [Be 2], Section 1, the $G$-action on $\mathbf{P}(X)$ lifts to a $G$-action on $X$. Moreover the above embedding lifts to a $G$-equivariant closed embedding $X \rightarrow \mathfrak{g}^*$. By this embedding $X$ is identified with a coadjoint orbit closure of $\mathfrak{g}^*$. In particular, $G$ acts transitively on $X_{\text{reg}}$. But we do not know when $G$ is semisimple.
References


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