Deformations of finite groups and hypergroups

Tatsuya Tsurii
Graduate School of Science, Osaka Prefecture University

Abstract

In this note, we introduce $q$-deformations of finite groups of low order, for examples, cyclic groups, symmetric groups, dihedral groups and the quaternion group in the category of hypergroups. Moreover we discuss $q$-deformations of certain finite hypergroups.

1. Introduction

We investigate $q$-deformations of finite groups and finite hypergroups in the category of hypergroup. It is known that there is no $q$-deformations of finite groups in the category of quantum groups. However we introduce that there are many $q$-deformations of finite groups in the category of hypergroups.

Hypergroups $Z_q(2)$ of order two with a parameter $q (0 < q \leq 1)$ are interpreted as $q$-deformations of the cyclic group $Z_2$. This fact is our motivation that we started to investigate $q$-deformations of finite groups and finite hypergroups.

In section 3, we discuss $q$-deformations of the cyclic group $Z_3$ of order three and the cyclic group $Z_4$ of order four. In section 4, we discuss $q$-deformations of the symmetric group $S_3$, the dihedral group $D_4$ and quaternion group $Q_4$. These $q$-deformations are given by applying a notion of a semi-direct product hypergroup introduced by H. Heyer and S. Kawakami (see [5]).

Moreover we study $q$-deformations of certain finite hypergroups of low order, the orbital hypergroups $K^a(Z_3)$ of $Z_3$ and $K^a(Z_4)$ of $Z_4$, the character hypergroups $K(S_3)$ of $S_3$, $K(D_4)$ of $D_4$ and $K(Q_4)$ of $Q_4$, the conjugacy class hypergroups $K(S_3)$ of $S_3$, $K(D_4)$ of $D_4$ and $K(Q_4)$ of $Q_4$ in section 5.

2. Preliminaries

For a finite set $K = \{c_0, c_1, \cdots, c_n\}$, we denote by $M^b(K)$ and $M^1(K)$, the set of all complex valued measures on $K$ and the set of all non-negative probability measures on $K$ respectively, namely

$$M^b(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} : a_j \in \mathbb{C} \ (j = 0, 1, 2, \cdots, n) \right\},$$

$$M^1(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} : a_j \geq 0 \ (j = 0, 1, 2, \cdots, n), \ \sum_{j=0}^{n} a_j = 1 \right\}$$

where the symbol $\delta_c$ stands for the Dirac measure at $c \in K$. For $\mu = a_0 \delta_{c_0} + a_1 \delta_{c_1} + \cdots + a_n \delta_{c_n} \in M^b(K)$, the support of $\mu$ is

$$\text{supp}(\mu) := \{ c_j \in K : a_j \neq 0 \ (j = 0, 1, 2, \cdots, n) \}.$$
**Axiom** A finite hypergroup \( K = (K, M^b(K), \circ, \ast) \) consists of a finite set \( K = \{c_0, c_1, \cdots, c_n\} \) together with an associative product (called convolution) \( \circ \) and an involution \( \ast \) in \( M^b(K) \) satisfying the following conditions.

1. The space \( (M^b(K), \circ, \ast) \) is an associative \(*\)-algebra with unit \( \delta_{c_0} \).
2. For \( c_i, c_j \in K \), the convolution \( \delta_{c_i} \circ \delta_{c_j} \) belongs to \( M^1(K) \).
3. There exists an involutive bijection \( c_i \mapsto c_i^* \) on \( K \) such that \( \delta_{c_i^*} = \delta_{c_i}^* \).

Moreover \( c_j = c_i^* \) if and only if \( c_0 \in \text{supp}(\delta_{c_i} \circ \delta_{c_j}) \) for all \( c_i, c_j \in K \).

A finite hypergroup \( K \) is called **commutative** if the convolution \( \circ \) on \( M^b(K) \) is commutative.

Let \( K \) and \( L \) be finite hypergroups. A mapping \( \varphi : K \to L \) is called a (hypergroup) **homomorphism of \( K \)** into \( L \) if there exists a \(*\)-homomorphism \( \hat{\varphi} \) of \( M^b(K) \) into \( M^b(L) \) as \(*\)-algebras such that \( \delta_{\varphi(c)} = \hat{\varphi}(\delta_c) \). If \( \hat{\varphi} \) is bijective, \( \varphi \) is called an **isomorphism** of \( K \) onto \( L \). In the case that \( L = K \), an isomorphism \( \varphi : K \to K \) is called an **automorphism** of \( K \). The set of all automorphisms of \( K \) becomes a group and it is denoted by \( \text{Aut}(K) \). Let \( G \) be a finite group. A homomorphism \( \alpha : G \to \text{Aut}(K) \) is called an action of \( G \) on \( K \).

For a commutative hypergroup \( K \), a complex-valued function \( \chi \) on \( K \) is called a **character** if \( \chi \) is linearly extendable on \( M^b(K) \) to be \( \hat{\chi}(\delta_{c_i}) = \chi(c_i) \) and satisfying that \( \hat{\chi}(\delta_{c_i} \circ \delta_{c_j}) = \hat{\chi}(\delta_{c_i}) \hat{\chi}(\delta_{c_j}) \) and \( \hat{\chi}(\delta_{c_i}^*) = \overline{\hat{\chi}(\delta_{c_i})} \) for all \( c_i, c_j \in K \).

We denote the trivial character by \( \chi_0 \). Let \( \hat{K} \) be the set of all characters of \( K \). A convolution on \( \hat{K} \) is defined by multiplication of functions on \( K \). Then \( \hat{K} \) becomes a signed hypergroup and the duality \( \hat{K} \cong K \) holds.

1. **Conjugacy class hypergroup** Let \( G \) be a finite group. For \( g \in G \), put \( \alpha_g(k) = Ad_g(k) = gkg^{-1} \) (\( k \in G \)). Then \( \alpha \) is an action of \( G \) on \( G \). Hence we obtain the orbital hypergroup \( \mathcal{K}^G(G) \) which we denote by \( \mathcal{K}(G) \) which is called a conjugacy class hypergroup of \( G \).

2. **Character hypergroup** For a finite group \( G \), \( \hat{G} = \{\pi_0, \pi_1, \cdots, \pi_m\} \) is the set of all equivalence classes of irreducible representations of \( G \). For \( \pi_j \in \hat{G} \), a character \( \chi_j \) associated with \( \pi_j \) is defined by

\[
\chi_j(g) = \frac{1}{\dim \pi_j} \text{tr}(\pi_j(g)).
\]

Then \( \mathcal{K}(\hat{G}) = \{\chi_0, \chi_1, \cdots, \chi_m\} \) becomes a commutative hypergroup with unit \( \chi_0 \) by the multiplication of functions on \( G \).

3. **Hypergroup join** For two finite hypergroups \( H = \{h_0, h_1, \cdots, h_m\} \) and \( L = \{\ell_0, \ell_1, \cdots, \ell_k\} \), a hypergroup join

\[
H \vee L = \{h_0, h_1, \cdots, h_m, \ell_1, \cdots, \ell_k\}
\]
is defined by the convolution $\diamond$ whose structure equations are

\[
\begin{align*}
\delta_{h_i} \circ \delta_{h_j} &= \delta_{h_i} \circ \delta_{h_j}, \\
\delta_{h_i} \circ \delta_{\ell_j} &= \delta_{\ell_i}, \\
\delta_{\ell_i} \circ \delta_{\ell_j} &= \delta_{\ell_i} \circ \delta_{\ell_j} \text{ when } \ell_j \neq \ell_i, \\
\delta_{\ell_i} \circ \delta_{\ell_j}^* &= n_i^0 \omega(H) + \sum_{j=1}^{k} n_i^j \delta_{\ell_j}
\end{align*}
\]

where $\delta_{\ell_i} \circ \delta_{\ell_j}^* = n_i^0 \delta_{\ell_0} + \sum_{j=1}^{k} n_i^j \delta_{\ell_j}$ and $\omega(H)$ is the normalized Haar measure of $H$.

3. Deformations of finite abelian groups

Let $K = \{c_0, c_1\}$ be a hypergroup of order two. Then the structure of $K$ is determined by

\[
\delta_{c_1} \circ \delta_{c_1} = q \delta_{c_0} + (1 - q) \delta_{c_1}
\]

where $0 < q \leq 1$. We denote it by $Z_q(2)$ which is interpreted as a $q$-deformation of $Z_2$. Stimulating by this fact, we have started to study $q$-deformations of finite groups.

3.1 Deformation $Z_q(3)$ of $Z_3$

First of all we discuss a $q$-deformation of $Z_3$. It is easy to check the following proposition directly and this fact is also described in the paper (Wildberger [21]).

**Proposition 3.1** Let $K = \{c_0, c_1, c_2\}$ be a hypergroup of order three. For each $q$ ($0 < q \leq 1$) there exists a unique hypergroup of order three such that $\delta_{c_1} = \delta_{c_2}$ and $\delta_{c_1} \circ \delta_{c_2} = q \delta_{c_0} + \cdots$.

We denote the above $K$ by $Z_q(3)$, which is interpreted as a $q$-deformation of $Z_3$. The structure equations of $Z_q(3) = \{c_0, c_1, c_2\}$ ($0 < q \leq 1$) are determined by

\[
\begin{align*}
\delta_{c_1} \circ \delta_{c_2} &= q \delta_{c_0} + \frac{1 - q}{2} \delta_{c_1} + \frac{1 - q}{2} \delta_{c_2}, \\
\delta_{c_1} \circ \delta_{c_1} &= \frac{1 - q}{2} \delta_{c_1} + \frac{1 + q}{2} \delta_{c_2}, \\
\delta_{c_2} \circ \delta_{c_2} &= \frac{1 + q}{2} \delta_{c_1} + \frac{1 - q}{2} \delta_{c_2}.
\end{align*}
\]

Put $\overline{Z_q(3)} = \{\chi_0, \chi_1, \chi_2\}$. Then the character table of $Z_q(3)$ is

<table>
<thead>
<tr>
<th>$\chi_0$</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$\omega_q$</td>
<td>$\overline{\omega_q}$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$\overline{\omega_q}$</td>
<td>$\omega_q$</td>
</tr>
</tbody>
</table>

where $\omega_q = \frac{-q + i \sqrt{q^2 + 2q}}{2}$. 

3
By the symmetry of the character table we see that $\widehat{\mathbb{Z}_4}(3) \cong \mathbb{Z}_4(3)$.

3.2 Deformation $\mathbb{Z}_{(p,q)}(4)$ of $\mathbb{Z}_4$

We investigated several kinds of extension problem in the category of commutative hypergroups, refer to [6], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The cyclic group $\mathbb{Z}_4$ of order four is a non-splitting extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Then one can consider a non-splitting extension $\mathbb{Z}_{(p,q)}(4)$ ($0 < p \leq 1$, $0 < q \leq 1$) of $\mathbb{Z}_4(2)$ by $\mathbb{Z}_p(2)$ as follows.

**Proposition 3.2** (Example 4.2 in [14]) For $(p, q)$ ($0 < p \leq 1$, $0 < q \leq 1$) there exists a unique hypergroup $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ of order four, which is an extension hypergroup of $\mathbb{Z}_4(2)$ by $\mathbb{Z}_p(2) = \{c_0, c_2\}$ such that $c_1^4 = c_3$.

The structure of $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ ($0 < p \leq 1, 0 < q \leq 1$) is given by

$$
\begin{align*}
\delta_{c_1} \circ \delta_{c_1} &= \delta_{c_3} \circ \delta_{c_3} = \frac{1 - q}{2} \delta_{c_1} + q \delta_{c_2} + \frac{1 - q}{2} \delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_2} &= p \delta_{c_0} + (1 - p) \delta_{c_2}, \quad \delta_{c_1} \circ \delta_{c_2} = \frac{1 - p}{2} \delta_{c_1} + \frac{1 + p}{2} \delta_{c_3}, \\
\delta_{c_1} \circ \delta_{c_3} &= \frac{2pq}{1 + p} \delta_{c_0} + \frac{1 - q}{2} \delta_{c_1} + \frac{q - pq}{1 + p} \delta_{c_2} + \frac{1 - q}{2} \delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_3} &= \frac{1 + p}{2} \delta_{c_1} + \frac{1 - p}{2} \delta_{c_3}.
\end{align*}
$$

Put $\mathbb{Z}_{(p,q)}(4) = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then the character table of $\mathbb{Z}_{(p,q)}(4)$ is

<table>
<thead>
<tr>
<th></th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$i\sqrt{pq}$</td>
<td>$-p$</td>
<td>$-i\sqrt{pq}$</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$-q$</td>
<td>1</td>
<td>$-q$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>$-i\sqrt{pq}$</td>
<td>$-p$</td>
<td>$i\sqrt{pq}$</td>
</tr>
</tbody>
</table>

It is easy to see that $\widehat{\mathbb{Z}_{(p,q)}(4)} \cong \mathbb{Z}_{(q,p)}(4)$ and $\mathbb{Z}_{(p,q)}(4)$ is interpreted as a $(p, q)$-deformation of $\mathbb{Z}_4$.

4. Deformations of non-abelian finite groups

Let $\alpha$ be an action of a finite group $G$ on a finite hypergroup $H = (H, M^b(H), \circ, \ast)$. Then a semi-direct product hypergroup $S := H \rtimes_\alpha G$ is introduced in [5]. A convolution $\varnothing$ in $M^b(S)$ is defined by

$$(\varepsilon_{h_1} \otimes \delta_{g_1}) \varnothing (\varepsilon_{h_2} \otimes \delta_{g_2}) := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)} \otimes \delta_{g_1 g_2}),$$

where $\varepsilon$ and $\delta$ stand for Dirac measures in $M^b(H)$ and $M^b(G)$ respectively. Unit element is $\varepsilon_e \otimes \delta_e$. An involution $-$ is

$$(\mu \otimes \delta_g)^{-} := \alpha_{g}^{-1}(\mu^{\ast}) \otimes \delta_{g^{-1}}$$

4
for all $\mu \in M^b(H)$ and $g \in G$.

4.1 Deformation $S_q(3)$ of the symmetric group $S_3$

The symmetric group $S_3$ is a semi-direct product $Z_3 \rtimes \alpha Z_2$ where $\alpha$ is an action of $Z_2$ on $Z_3$.

Let $\alpha$ be an action of $Z_2 = \{e, g\}$ on a hypergroup $Z_q(3) = \{h_0, h_1, h_2\}$ ($0 < q \leq 1$) such that

$$\alpha_g(h_1) = h_2, \quad \alpha_g(h_2) = h_1.$$ 

Then we obtain a semi-direct product hypergroup

$$S_q(3) := Z_q(3) \rtimes \alpha Z_2$$

which is a $q$-deformation of the symmetric group $S_3 = Z_3 \rtimes \alpha Z_2$.

4.2 Deformation $D_{(p,q)}(4)$ of the dihedral group $D_4$

The dihedral group $D_4$ is written by a semi-direct product $Z_4 \rtimes \alpha Z_2$.

Let $H = Z_{(p,q)}(4) = \{h_0, h_1, h_2, h_3\}$ ($0 < p \leq 1, 0 < q \leq 1$) be the $(p, q)$-deformation of $Z_4$ and $\alpha$ an action of $Z_2 = \{e, g\}$ on $Z_{(p,q)}(4)$ given by

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$ 

Then we obtain a semi-direct product hypergroup

$$D_{(p,q)}(4) := Z_{(p,q)}(4) \rtimes \alpha Z_2.$$ 

Hence, we obtain a $(p, q)$-deformation $D_{(p,q)}(4)$ of the dihedral group $D_4$.

4.3 Another deformation $W_q(4)$ of the dihedral group $D_4$

The dihedral group $D_4$ is also written by a semi-direct product $(Z_2 \times Z_2) \rtimes \beta Z_2$ where $\beta$ is a flip action of $Z_2$ on $Z_2 \times Z_2$.

Let $Z_q(2) \times Z_q(2) = \{(h_0, h_0), (h_0, h_1), (h_1, h_0), (h_1, h_1) ; h_0, h_1 \in Z_q(2)\}$ be a $q$-deformation of $Z_2 \times Z_2$. Let $\beta$ be a flip action of $Z_2 = \{e, g\}$ on $Z_q(2) \times Z_q(2)$ given by

$$\beta_g((h_i, h_j)) = (h_j, h_i) \quad (i, j = 0 \text{ or } 1).$$ 

Then we obtain a semi-direct product hypergroup

$$W_q(4) := (Z_q(2) \times Z_q(2)) \rtimes \beta Z_2.$$ 

The hypergroup $W_q(4)$ is another $q$-deformation of $D_4$.

4.4 Deformation $Q_q(4)$ of the quaternion group $Q_4$

The structure of the quaternion group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is determined by

$$i^2 = j^2 = k^2 = -1, \quad ij = k.$$
Let $\alpha$ be an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_4 = \{h_0, h_1, h_2, h_3\}$ such that $\alpha_g(h_1) = h_3, \; \alpha_g(h_2) = h_2, \; \alpha_g(h_3) = h_1$.

Let $c$ be a $H$-valued 2-cocycle of $\mathbb{Z}_2$ which is also given by $c(e, e) = c(e, g) = c(g, e) = h_0$ and $c(g, g) = h_2$.

Then a twisted semi-direct product group $\mathbb{Z}_4 \rtimes^c_\alpha \mathbb{Z}_2$ is defined by the product
\[(h, g)(h', g') = (h\alpha_g(h')c(g, g'), gg')\]
for $h, h' \in \mathbb{Z}_4$ and $g, g' \in \mathbb{Z}_2$. The quaternion group $Q_4$ is isomorphic to $\mathbb{Z}_4 \rtimes^c_\alpha \mathbb{Z}_2$.

Let $H = \mathbb{Z}_{(1,q)}(4) = \{h_0, h_1, h_2, h_3\}$ be a $q$-deformation of $\mathbb{Z}_4$. Then, we obtain a twisted semi-direct product hypergroup
\[Q_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes^c_\alpha \mathbb{Z}_2.\]
The hypergroup $Q_q(4)$ is a $q$-deformation of the quaternion group $Q_4 = \mathbb{Z}_4 \rtimes^c_\alpha \mathbb{Z}_2$.

5. Deformations of finite hypergroups

In this section we discuss $q$-deformations of several kinds of finite hypergroups in a similar way to the case of finite groups.

5.1 Deformations of orbital hypergroups

(4) Orbital hypergroup Given an action $\alpha$ of a finite group $G$ on a commutative hypergroup $H$, we obtain a orbit $O = \{\alpha_g(h) : g \in G\}$ of $h \in H$ under the action $\alpha$. Let $\{O_0, O_1, \cdots, O_m\}$ be the set of all orbits in $H$. We denote an element $c_j$ which is corresponding to each orbit $O_j$ and put $H^\alpha = \{c_0, c_1, \cdots, c_m\}$. Let $M^b(H)^\alpha$ denote the fixed point algebra of $M^b(H)$ under the action $\alpha$, namely
\[M^b(H)^\alpha = \{\mu \in M^b(H) : \alpha_g(\mu) = \mu \text{ for all } g \in G\}.\]
We note that $M^b(H)^\alpha$ is a $\ast$-subalgebra of $M^b(H)$. For $c_j \in H^\alpha$, put
\[\delta_{c_j} = \frac{1}{|O_j|} \sum_{h \in O_j} \delta_h = \frac{1}{|G|} \sum_{g \in G} \alpha_g(\delta_h).\]
Then $\delta_{c_j} \in M^b(H)^\alpha \cap M^1(H)$. $K^\alpha(H) = (H^\alpha, M^b(H)^\alpha, \circ, \ast)$ becomes a hypergroup which is called an orbital hypergroup of $H$ by the action $\alpha$.

Example 1 The orbital hypergroup $K^\alpha(\mathbb{Z}_q(3)) = \{c_0, c_1\}$ is a $q$-deformation of $K^\alpha(\mathbb{Z}_3)$. 
The structure equations are

\[ \delta c_1 \circ \delta c_1 = \frac{q}{2} \delta c_0 + \left(1 - \frac{q}{2}\right) \delta c_1. \]

**Example 2** The orbital hypergroup \( K^\alpha(\mathbb{Z}_{(p,q)}(4)) = \{c_0, c_1, c_2\} \) is a \( q \)-deformation of \( K^\alpha(\mathbb{Z}_4) \).

The structure equations are

\[ \delta c_1 \circ \delta c_1 = p \delta c_0 + (1 - p) \delta c_1, \quad \delta c_1 \circ \delta c_2 = \delta c_2, \]
\[ \delta c_2 \circ \delta c_2 = \frac{pq}{1 + p} \delta c_0 + \frac{q}{1 + p} \delta c_1 + (1 - q) \delta c_2. \]

### 5.2 Deformations of character hypergroups

(5) **Character hypergroup of semi-direct product hypergroup** Let \( S = H \rtimes_\alpha G \) be a semi-direct product hypergroup defined by an action \( \alpha \) of a finite abelian group \( G \) on a finite commutative hypergroup \( H \) (Refer to [5]). \( \hat{S} = \hat{H} \rtimes_\alpha \hat{G} \) is the set of all equivalence classes of irreducible representations of \( S \).

For \( (\pi, \mathcal{H}(\pi)) \in \hat{S} \), the character \( ch(\pi) \) of \( \pi \) is defined by

\[ ch(\pi)((h, g)) = \frac{1}{\dim \pi} \text{tr}(\pi(h, g)) \]

where \( (h, g) \in H \rtimes_\alpha G \) and \( \text{tr} \) is the trace of \( B(\mathcal{H}(\pi)) \) (Refer to [K,Y]). Put \( \mathcal{K}(\hat{S}) = \{ch(\pi) ; \pi \in \hat{S}\} \).

**Proposition 5.1** ([7]) If the action \( \alpha \) satisfies the regularity condition, then \( \mathcal{K}(\hat{H} \rtimes_\alpha \hat{G}) \) becomes a commutative hypergroup by the product of functions on \( S = H \rtimes_\alpha G \).

This hypergroup is called a character hypergroup of the semi-direct product hypergroup \( S = H \rtimes_\alpha G \).

**Example 3** The character hypergroup \( \mathcal{K}(\hat{S}_q(3)) \) of \( S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2 \) is a \( q \)-deformation of \( \mathcal{K}(\hat{S}_3) \).

**Remark** we note that \( \mathcal{K}(\hat{S}_q(3)) = \mathbb{Z}_2 \vee \mathbb{Z}_q(2) \).

**Example 4** The character hypergroup \( \mathcal{K}(\hat{D}_{(p,q)}(4)) \) of \( D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2 \) is a \( q \)-deformation of \( \mathcal{K}(\hat{D}_4) \).

**Example 5** The character hypergroup \( \mathcal{K}(\hat{Q}_{q}(4)) \) of \( Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2 \) is a \( q \)-deformation of \( \mathcal{K}(\hat{D}_4) \).

**Remark** We see that \( \mathcal{K}(\hat{Q}_q(4)) = \mathcal{K}(\hat{D}_{(1,q)}(4)) \) although \( Q_q(4) \) is not isomorphic to \( D_{(1,q)}(4) \).

### 5.3 Deformations of conjugacy class hypergroups

7
(6) **Generalized conjugacy class hypergroup** Let \( H \rtimes_{\alpha} G \) be a semi-direct product hypergroup. Then there exists the canonical conditional expectation \( E \) from \( M^b(S) \) onto the center \( Z(M^b(H \rtimes_{\alpha} G)) \) of \( M^b(H \rtimes_{\alpha} G) \). Put

\[
\mathcal{K}(H \rtimes_{\alpha} G) = \{ E(\delta_{(h,g)}) ; (h,g) \in H \rtimes_{\alpha} G \}.
\]

**Proposition 5.2** ([6]) If the action \( \alpha \) satisfies the regularity condition, then \( \mathcal{K}(H \rtimes_{\alpha} G) \) becomes a commutative hypergroup with the convolution in \( Z(M^b(H \rtimes_{\alpha} G)) = M^b(\mathcal{K}(H \rtimes_{\alpha} G)) \). Moreover \( \mathcal{K}(H \rtimes_{\alpha} G) \cong \mathcal{K}(H \rtimes_{\alpha} G) \) holds.

We call \( \mathcal{K}(H \rtimes_{\alpha} G) \) a generalized conjugacy class hypergroup of \( H \rtimes_{\alpha} G \).

**Example 6** The conjugacy class hypergroup \( \mathcal{K}(S_q(3)) \) of \( S_q(3) \) is a \( q \)-deformation of \( \mathcal{K}(S_3) \).

**Remark** We note that \( \mathcal{K}(S_q(3)) = \mathbb{Z}_q(2) \vee \mathbb{Z}_2 \).

**Example 7** Conjugacy class hypergroup \( \mathcal{K}(D_{p,q}(4)) \) of \( D_{p,q}(4) \) is a \((p,q)\)-deformation of \( \mathcal{K}(D_4) \).

**Example 8** The conjugacy class hypergroup \( \mathcal{K}(Q_q(4)) \) of \( Q_q(4) \) is a \( q \)-deformation of \( \mathcal{K}(Q_4) \).

**Remark** We note that \( \mathcal{K}(D_{1,q}) \cong \mathcal{K}(Q_q(4)) \).

**References**


[7] Ito, W. and Kawakami, S. : Crossed Products of Commutative Finite Hyper-


**Addresses**

Tatsuya Tsurii : Graduate School of Science, Osaka Prefecture University
1-1 Gakuen-cho, Nakaku, Sakai
Osaka, 599-8531
Japan
e-mail : dw301003@edu.osakafu-u.ac.jp